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# Climate, Black Holes and Vorticity: How on *Earth* are They Related?

*By George Haller*

In short, they are related through oceanic eddies. Often called the weather of the ocean, eddies are gigantic vortices of swirling water. While they exist across a range of spatial scales, perhaps most fascinating are mesoscale eddies, varying between 100 and 200 km in diameter. These eddies are too large to recognize from a ship or an airplane, but were too small to be visible in early satellite observations of the ocean. It wasn't until the 1960s that they were first recorded due to improved satellite altimetry.

Coherent mesoscale eddies, which keep their integrity for extended times, are envisioned to carry the same water body without substantial leakage and deformation. Coherent fluid transport in the unsteady ocean, however, is not directly observable, and thus one must rely on sporadic observations of transported scalars, such as chlorophyll and temperature, to gain insight into material currents. Some exceptional chlorophyll patches captured by eddies drift in the ocean for up to a year or more (see Figure 1a). The carrier eddies show no substantial mixing with surrounding waters, often creating moving oases for the marine food chain.

### Why and How to Track Eddies?

Amidst concerns over climate change, episodic observations of material transport in the ocean are insufficient, and more quantitative and reliable eddy identification tools are needed. For instance, the Agulhas rings, the largest mesoscale eddies in the ocean (see Figure 1b), are believed to transport warm and salty water from the Indian Ocean across the South Atlantic through the so-called Agulhas leakage, which is reportedly on the rise [3]. The rings might traverse as far as the upper arm of the Atlantic Meridional Overturning Circulation (AMOC), whose potential slowdown due to melting sea ice in a warming climate is of current concern. This rise leads to the generation of more Agulhas rings, possibly weakening the AMOC slowdown [1]. To assess the validity of such hypotheses, one must uncover the exact coherent cores of the Agulhas rings from available observational velocity data.

*See* **Black Holes** *on page 3*

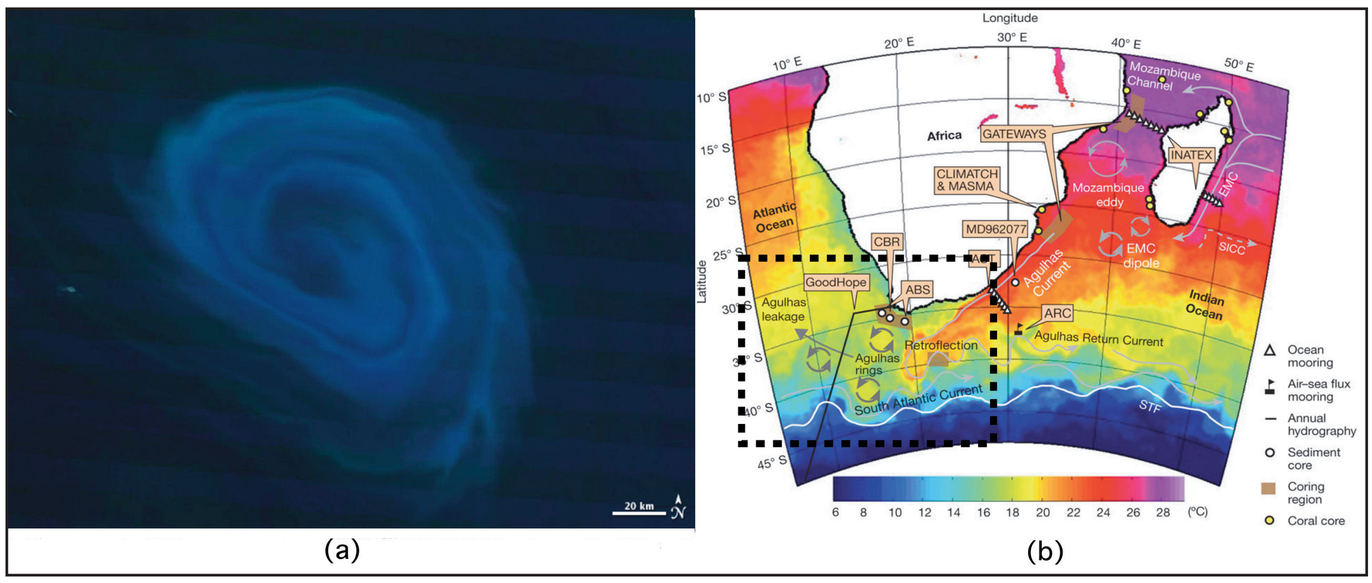


**Figure 1a.** *Satellite image of a chlorophyll patch captured by an Agulhas ring. Image credit: NASA Earth Observatory/Jesse Allen.* **1b.** *A sketch of the Agulhas leakage. Adapted with permission from Macmillan Publishers Ltd.: Nature [1], copyright (2011).*

# Tracing Genealogy Within an Invasion Wave

*By Kerry Landman*

Invasion waves arise in many systems, including wound healing, brain tumor expansion, and the displacement of indigenous species by an introduced species.

Mathematical models of invasion systems are often described by Fisher's equation, which contains two essential mechanisms – the ability of the entities to move and increase in number through proliferation (cell division or reproduction). Fisher's equation contains a diffusion term and a logistic growth term, which includes crowding effects involving a carrying capacity:

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} + \lambda C\left(1 - \frac{C}{K}\right). \qquad (1)$$

Phase plane arguments demonstrate that the partial differential equation (PDE) exhibits a travelling wave solution with a minimum wavespeed $2\sqrt{D\lambda}$. Entities behind the wavefront are at carrying capacity, so proliferation only occurs in the region of the wavefront, causing the travelling wave to steadily advance. We call this *frontal expansion.* Such models are used to study population-level properties of invasion.

### Large Individual Variability Within the Predictable Travelling Wave

The enteric nervous system (ENS) is a large complex system—comparable in size to the spinal cord—in the wall of the gastrointestinal tract. It is responsible for normal gut function and peristaltic contraction, which forces food along the gastrointestinal tract. During vertebrate embryonic ENS development, a small population of immigrant neural crest cells (NCC) enters the stomach and progressively invades and colonizes the small and large intestine as a travelling wave over many days (mouse, chick), and weeks (human). The cells actively move in a two-dimensional cylindrical shell within the gut wall and undergo many rounds of cell division (without cell death). Our group at the University of Melbourne has a decade-long collaboration with Don Newgreen's Embryology Laboratory at the Murdoch Children's Research Institute on this ENS cell invasion process.

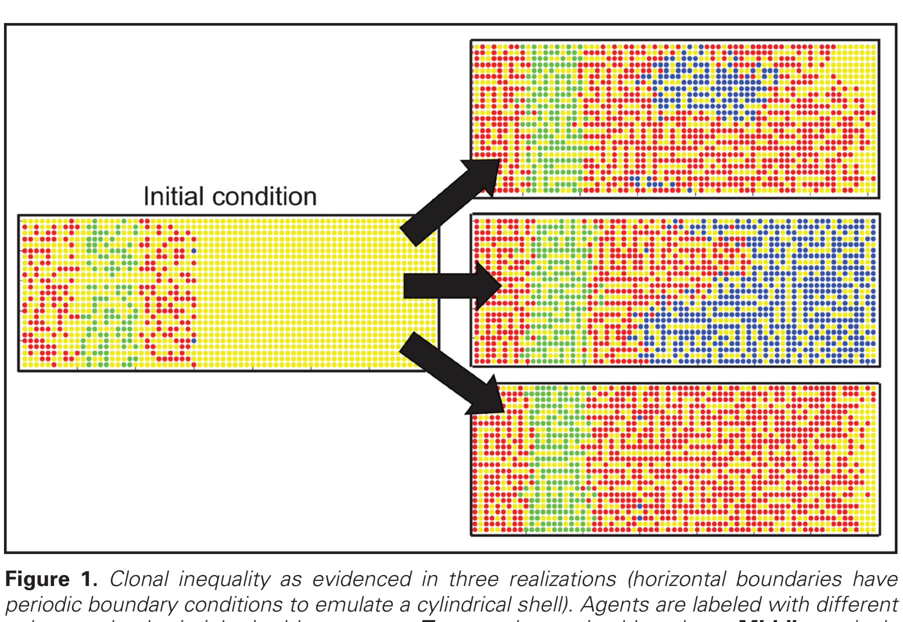


**Figure 1.** *Clonal inequality as evidenced in three realizations (horizontal boundaries have periodic boundary conditions to emulate a cylindrical shell). Agents are labeled with different colors, and color is inherited by progeny.* ***Top:*** *moderate-size blue clone.* ***Middle:*** *a single blue agent has the large clone, responsible for invasion wave.* ***Bottom:*** *insignificant clone sizes. Image credit: B.J. Binder.*

The evolution of the NCC population density is well-described by Fisher's equation. In addition, agent-based models facilitate the probing of more detailed information on cell-cell interaction. We use a square lattice agent-based model for our NCC invasion system, where an agent represents a single NCC. One agent at most can occupy a lattice site at each timestep, defining an exclusion process. We then assign probabilities to local rules describing cell motility and proliferation. Using these simple simulation rules, a single realization produces a right-moving wave (with seeding agents on the left). Averaging over many simulations leads to a predictable travelling wave, analogous to Fisher's wave.

We label agents with different colors to observe the interactions between various parts of the invasion wave. Progeny inherit the color of the parent agent, and all agents follow the same local rules. Every realization is slightly different, but at the population level each demonstrates frontal expansion as a result of progeny from the red and blue agents (see Figure 1).

In these models, we label every starting agent and determine the genealogy. The clonal contribution of each agent is highly variable, from minimal contribution to a few clones of overwhelming size, which we term *superstars*. We set out to determine how common these behaviors are, and if they could be shown experimentally.

Superstars are apparent in every realization (see Figure 2, on page 4). This behavior is the result of stochastic competition for space. All agents have the same ability to move and divide, but most become blocked by surrounding agents and thus can no longer do so. There is nothing inherently different about superstars. They just got lucky.

Using 'cloning-in-a-crowd' experiments in gut explants, Dr. Newgreen's lab showed that in a crowd of unlabeled NCC, a single cell lineage can be traced using a green marker inherited in cell division. After several days, the whole gut is full of NCC.

*See* **Invasion Wave** *on page 4*

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# Obituaries

*By Mihalis Yannakakis and Cliff Stein*

David Stifler Johnson, a leader in theoretical computer science, passed away on March 8, 2016, at the age of 70. David is well known for his fundamental contributions to NP-completeness theory, optimization and approximation algorithms, and the experimental analysis of algorithms.

Born December 9, 1945, David received a B.A. in mathematics from Amherst College in 1967 and an M.S. in mathematics from the Massachusetts Institute of Technology (MIT) in 1968. After serving in the U.S. Army for two years, he returned to MIT, where he obtained his Ph.D. in 1973. In his Ph.D. thesis, "Near-Optimal Bin Packing Algorithms," David analyzed the performance of efficient bin packing algorithms, proving tight results on the proximity of the solutions they compute to the optimal solutions.

After his graduation, David joined AT&T Bell Laboratories, where he worked in the Mathematical Sciences Research Center from 1973 until 1996. He was head of the Mathematical Foundations of Computing Department from 1988 onwards. When AT&T split in 1996, David joined AT&T Labs Research, serving as head of the Algorithms and Optimization Research Department until 2013. He then joined the faculty of Columbia University as a visiting professor in the Department of Computer Science.

David was a major contributor to the early development of the theory and applications of NP-completeness. This theory established a close relationship between many seemingly-intractable problems that arise in a diverse range of fields and are believed to be unsolvable in polynomial time. David (and his collaborators) showed the NP-completeness of many basic problems from a variety of areas and introduced central concepts, e.g. the notions of strong NP-completeness and pseudo-polynomial algorithms, that identify more specifically the source of intractability. However, David's greatest impact in this field is through his book, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, coauthored by Michael R. Garey and published in 1979. In addition to presenting the fundamentals of NP-completeness theory, the book contains an extensive, systematic compendium of NP-complete problems known at the time, making it an invaluable reference. It has served generations of computer scientists and engineers as a handbook for differentiating computational problems that are solvable by practical, efficient methods from those that are not, and is one of the most cited references in all of computer science. A year after its publication, David and Garey received the Operations Research Society of America's Lanchester Prize.

Beginning in 1982, David wrote "The NP-completeness Column: an Ongoing Guide," a regular series of articles in the *Journal of Algorithms* and later *The ACM Transactions on Algorithms*, which covered new developments in the theory of NP-completeness and in related areas of algorithms and complexity theory.

David's early work was also instrumental in laying the foundation for the theory of approximation algorithms, efficient algorithms that compute nearly optimal solutions to hard optimization problems. Starting with his Ph.D. thesis, he designed and analyzed approximation algorithms for many important problems, including packing and scheduling. David's 1973 paper, "Approximation Algorithms for Combinatorial Problems," is especially significant. In this paper, he studied central combinatorial problems like clique, graph coloring, set cover, and maximum satisfiability. He noted that, although these problems are polynomially equivalent in terms of reaching an optimal solution, they are very different with respect to efficiently finding good approximate solutions; this raised the question of how to classify the approximability of these and other hard combinatorial optimization problems. The area of approximation algorithms has flourished since then, building on the work of David and others, and continues to be one of the most active fields in theoretical computer science. The questions raised in David's 1973 paper eventually led to the development of the deep theory of probabilistically-checkable proofs some 20 years later, demonstrating that his algorithms were essentially optimal.

David Stifler Johnson, 1945-2016. Photo credit: http://davidsjohnson.net/.

David also played a prominent role in laying out rigorous foundations for the experimental analysis of algorithms. In a series of highly influential papers in the late 80s and 90s, he performed a comprehensive experimental analysis of different algorithms for important, well-studied problems, most notably the famous travelling salesman problem, and of general methodologies like simulated annealing. His papers are emblematic of rigor and thoroughness, and set the standard for this kind of work. David also initiated and led a series of annual Implementation Challenges at the Center for Discrete Mathematics and Theoretical Computer Science (DIMACS) that fostered rigorous experimental research in the algorithms community.

Besides his personal research, David was heavily involved in the theoretical computer science community, and has greatly impacted the field's growth. In 1990, he founded the ACM-SIAM Symposium on Discrete Algorithms (SODA), a top venue for research in the field, and served as steering committee chair for 25 years. David organized and chaired many other conferences, including the Federated Computing Research Conference and the ACM Symposium on Theory of Computing. He led the ACM Special Interest Group on Algorithms and Computation Theory (SIGACT) from 1987 to 1991, initiated several awards, chaired and served on many ACM and SIAM committees, and continued to impact the theoretical computer science community throughout his career. In recognition of his extraordinary contributions, David received the first SIGACT Distinguished Service Prize for his "selfless dedication and personal initiative in serving the Computer Science Theory community."

Numerous honors and prizes commemorate David's research. He was named a SIAM Fellow in 2009, and was also recognized as an ACM Fellow and an AT&T Fellow. David received the Donald E. Knuth Prize for his contributions to the theoretical and experimental analysis of algorithms, and was elected to the National Academy of Engineering for his "contributions to the theory and practice of optimization and approximation algorithms."

David was a very generous, humble, and modest person. He cared deeply about people, including the members of his department, his colleagues, and researchers from all over the world who often contacted him with questions on NP-completeness, experimental analysis, purported proofs of P=NP or P≠NP, and other topics. David was a great mentor to young researchers, including the many students that interned in the Labs during his career. He advocated for and advanced the careers of countless members of the community, and kept a list of young researchers who should be chosen for conference program committees. He was an inspiration to all of his colleagues and will be greatly missed.

*Mihalis Yannakakis is Hudson Professor of Computer Science at Columbia University. He was a friend and colleague of David at Bell Labs for many years. Cliff Stein is a professor of computer science and industrial engineering and operations research at Columbia University. He was David's summer intern in 1991; David was a mentor, colleague, and friend ever since.*

## Errata and Clarifications

**Missed Photo Credit:**
In the article entitled "Seven Decades of Mathematics and Mechanics" by Maria-Carme Calderer and Richard James, which appeared in the April 2016 issue of *SIAM News* (Volume 49, Number 3), the photo of Maria-Carme Calderer, Jerry Ericksen, and Irene Fonseca was kindly provided by David Kinderlehrer.

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# Black Holes

*Continued from page 1*

Eddy tracking methods used in oceanography are Eulerian in nature, devised to highlight features of the instantaneous surface velocity field $v(x,t)$. The same techniques are nevertheless also broadly believed to identify regions of elliptic (or vortical) trajectories $x(t)$ for the ordinary differential equation (ODE) $\dot{x}=v(x,t)$ [4]. Classic examples in the theory of non-autonomous ODEs show that such an inference is generally incorrect, even if $v$ is just linear in the spatial variable $x\in\mathbb{R}^2$. Indeed, it is simple to construct spatially linear unsteady solutions of the Navier–Stokes equation that are pronounced coherent vortices by all instantaneous Eulerian criteria, yet the norm of their trajectories grows exponentially in time [7]. The same effect causes Eulerian eddy tracking methods to overestimate coherent eddy transport in the Agulhas leakage by about an order of magnitude [2].

### Where do Black Holes Come In?

For unambiguous identification of perfectly coherent material vortices, one first needs a mathematical definition of their boundaries. Such a Lagrangian boundary should show no filamentation over a finite time interval $[t_0,t]$, in contrast to the intense and inhomogeneous deformation of material surfaces in turbulent waters outside the vortices. Using elementary continuum mechanics, one finds that the parametrized initial position $x_0(s)$ of such a coherent boundary must be a closed stationary curve of the averaged relative stretching functional

$$Q_{t_0}^t(\gamma)=\frac{1}{\sigma}\int_0^{\sigma}\frac{\sqrt{\langle x_0'(s),C_{t_0}^t(x_0(s))x_0'(s)\rangle}}{\sqrt{\langle x_0'(s),x_0'(s)\rangle}}\mathrm{d}s,$$

with $C_{t_0}^t=\left[\nabla F_{t_0}^t\right]^T\nabla F_{t_0}^t$ denoting the right Cauchy–Green strain tensor [6]. An argument utilizing Noether's theorem then implies that closed stationary curves of $Q_{t_0}^t$ are precisely the closed null-geodesics of the Lorentzian metric tensor family $E_\lambda(x_0)=\frac{1}{2}[C_{t_0}^t(x_0)-\lambda^2 I]$, parametrized by $\lambda\in\mathbb{R}^+$ [9].

This result reveals a surprising mathematical analogy between black holes in general relativity and vortices in the two-dimensional ocean [10]. In the former setting, a photon sphere is a nowhere space-like hypersurface of null-geodesics in space-time, with space-like projections that trap photons orbiting around a black hole forever [5]. In the context of the two-dimensional oceanic space-time, vortex boundaries take the role of such photon spheres. This then implies [9] that a metric singularity of $E_\lambda(x_0)$ must necessarily arise inside oceanic eddies (see Figure 2), just as metric singularities are believed to arise invariably inside black holes. So, at the level of a mathematical analogy, a material vortex to a two-dimensional ocean matches what a black hole is to Einstein's four-dimensional space-time.

Beyond providing a curious analogy, metric singularities of the generalized Green–Lagrange strain tensor $E_\lambda$ form the cornerstone of automated Lagrangian vortex detection schemes for large ocean data sets [13]. Figure 2a shows the evolution of black-hole type vortices in the South Atlantic, computed as null-geodesics encircling metric singularities of $E_\lambda$ [10]. The flow map $F_{t_0}^{t_0+135\,days}$ is computed by integration from a satellite-altimetry-based surface velocity field $v(x,t)$.

### Aren't Vortices Supposed to be Related to Vorticity?

They are, but there is a caveat. An important axiom of continuum physics is that material behavior, including material transport by vortices, cannot depend on the observer describing the behavior.

Thus, a self-consistent definition of material eddies must be invariant under all Euclidean observer changes of the form $x=R(t)\tilde{x}+b(t)$, where $R(t)$ is a proper orthogonal tensor family and $b(t)$ is an arbitrary translation family. While the functional $Q_{t_0}^t$ defining black-hole eddies is objective, the vorticity $\omega(x,t)=\nabla\times v(x,t)$ is not. Indeed, an observer change gives the transformed vorticity

$$\tilde{\omega}(\tilde{x},t)=R^T(t)\omega(x,t)+\dot{r}(t),$$

with the vector $\dot{r}$ denoting the angular velocity of the frame rotation induced by $R(t)$. Because of this $\dot{r}$ term, the vorticity vector fails to transform properly, as a vector under a linear operator $R$ would. For this reason, vorticity has long been absent from the toolkit of objective Lagrangian [7] and even Eulerian [12] coherent structure detection.

A recent extension of the classic polar decomposition to non-autonomous processes, however, reveals an intrinsic connection between vorticity and objective material rotation. Valid in any finite dimensions, the *dynamic polar decomposition* theorem [8] guarantees a unique factorization of the flow gradient as

$$\nabla F_{t_0}^t=\Phi_{t_0}^t\Theta_{t_0}^t M_{t_0}^t,$$

where the *dynamic stretch tensor* $M_{t_0}^t$ is the flow gradient of a purely straining flow; the *mean rotation tensor* $\Theta_{t_0}^t$ is the flow gradient of a spatially uniform rigid-body rotation; and the *relative rotation tensor* $\Phi_{t_0}^t$ is the flow gradient of the local deviation from that mean rotation. The material rotation angle generated by $\Phi_{t_0}^t$ about its time-varying axis of rotation turns out to be a frame-invariant quantity. This objective rotation angle is surprisingly simple to compute: it is given by the Lagrangian-Averaged Vorticity Deviation (LAVD),

$$\mathrm{LAVD}_{t_0}^t(x_0)=\int_{t_0}^t\left|\omega(x(s;x_0),s)-\bar{\omega}(s)\right|ds,$$

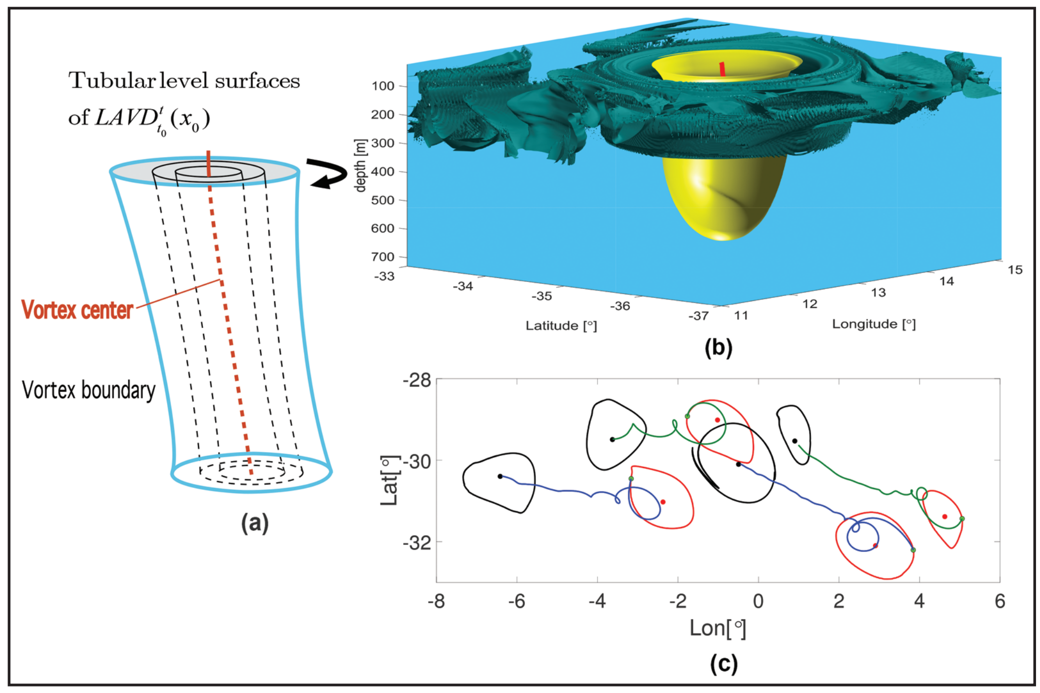


**Figure 3a.** *Rotationally coherent vortex boundary and center defined from the LAVD. Image credit: George Haller.* **3b.** *A three-dimensional Agulhas eddy boundary and eddy center (details in [11]).* **3c.** *Initial (red) and final (black) advected positions of LAVD-based Agulhas eddies over a period of three months. Floating objects (blue) converge to the centers of anti-cyclonic eddies; sinking objects (green) converge to the centers of cyclonic eddies (details in [11]). See online article (at sinews.siam.org) for animation. Image credit for 3b and 3c: Alireza Hadjighasem, ETH Zürich.*

with $\bar{\omega}(t)$ denoting the spatial mean of the vorticity [11].

Defining rotationally-coherent eddy boundaries as surfaces evolving from outermost tubular level sets of the LAVD provides the long-sought link between objective material eddies and vorticity. Unlike black-hole vortices, LAVD-based vortices may exhibit small tangential filamentation in their boundaries (see Figure 3a). The filaments, however, are bound to rotate with the material vortex without large-scale fingering into the surrounding turbulent waters. Figure 3b shows a three-dimensional example of this, with the velocity field generated by the Southern Ocean State Estimate (SOSE) model [14]. Material advection of this remarkably detailed material vortex boundary confirms the rotational coherence guaranteed by the *dynamic polar decomposition*.[1]

Remarkably, singular level sets at the core of nested tubular LAVD levels define vortex centers that can be proven to coincide exactly with the observed cyclonic repellers and anti-cyclonic attractors for positively buoyant inertial particles [11]. Hence, LAVD-based eddy centers are precisely the mysterious drifting locations that collect floating debris in the ocean. Figure 3c shows a numerical verification of this analytic prediction.

Implementing these mathematical advances in *in situ* analysis of the ocean and the atmosphere is an exciting perspective. Beyond quantifying mesoscale eddy transport, black-hole and LAVD eddies of smaller scales could aid real-time decision making in environmental disasters (e.g. oil spills) or in search and rescue operations. On the other extreme of the eddy scale spectrum, these techniques offer a frame-indifferent identification of gigantic material vortices in the atmospheres of other planets, such as Jupiter's Great Red Spot [6]. The quest to uncover coherent oceanic eddies has already lead to unexpected links to Lorentzian geometry and continuum mechanics, both of which deserve further exploration.

[1] See online article (at sinews.siam.org) for animation

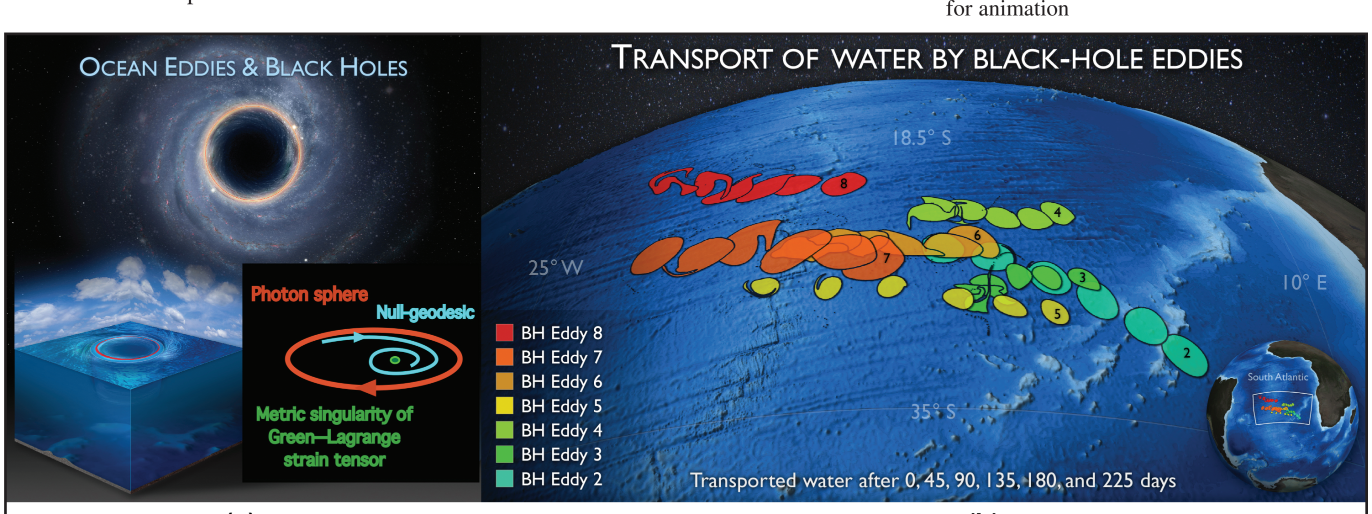


**Figure 2a.** *Closed null-geodesics of the two-dimensional generalized Green-Lagrange strain tensor are analogous to photon spheres in the four-dimensional space-time.* **2b.** *Materially advected coherent black-hole eddies in the Southern Ocean, identified from satellite data ranging over 135 days (details in [9]). See online article (at sinews.siam.org) for animation. Image credit: George Haller and the Center for Environmental Visualization, University of Washington.*

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## Invasion Wave

*Continued from page 1*

While most experiments had very few green-labeled cells, one showed a sea of green cells. This is the biological representative of a superstar – it is the cell that yields a disproportionately large contribution to the final population. These experiments show that clonal inequality is a reality, and that superstars do exist.

It is impossible to image all cells in the invading gut system, even with multicolor techniques. Hence, we can only trace a single lineage and collect data at a single time point in the gut experiments, unlike our agent-based models. However, researchers are currently developing techniques that could provide information on the number of cells in each generation.

In our agent-based models, we can color code the agents in the same generation, instead of visualizing individual lineages. Starting with generation-zero agents (as in Figure 2a), even a single realization shows that organization occurs within the spatial distribution of generation number during invasion, despite the large spatial variability of individual lineages. Can this organization be described with PDEs?

### PDEs Describing Generation Number Density

We developed a new system of PDEs to describe each cell generation number [2], which we derived from probability arguments and mean-field approximations. We considered how the average occupancy of each generation at a lattice site changes over a single timestep, while accounting for agents that leave and enter the site and noting that agents can only move into unoccupied sites. The mean-field approximation assumes that the occupancy status of neighboring sites is independent. Taking Taylor series expansions and the continuum limit leads to a coupled system of PDEs describing the generation number density $n_i(x,t)$ in terms of the total density $C$:

$$\frac{\partial n_i}{\partial t} = D\frac{\partial}{\partial x}\left((1-C)\frac{\partial n_i}{\partial x} + n_i\frac{\partial C}{\partial x}\right) + \lambda(2n_{i-1} - n_i)(1-C),$$

$$\frac{\partial n_0}{\partial t} = D\frac{\partial}{\partial x}\left((1-C)\frac{\partial n_0}{\partial x} + n_0\frac{\partial C}{\partial x}\right) - \lambda n_0(1-C).$$

Terms with $(1-C)$ arise due to the exclusion process, and the convective term arises due to the multi-species nature of the system. Of course, summing over all the generations gives Fisher's equation (1). Solving this numerically, we obtain the spatial distribution of generation number density; it increases in an organized manner from left to right (see Figure 3).

### Can the PDEs Predict Superstars?

From the PDE solutions, is it possible to predict lineage variability and the existence of superstars? To do so, we consider the flow of cells from one generation to the next at time $t$. From the spatial distributions, we know the number of cells in each generation at time $t$. Mass balance arguments determine a relationship between the number of cells and the number of cells that have undergone division in each generation. An explicit formula for the generation transition probabilities is obtained in terms of the number of cells in each generation, which allows us to define a Galton-Watson process, appropriate for cell division [2], and subsequently generate cell lineage data.

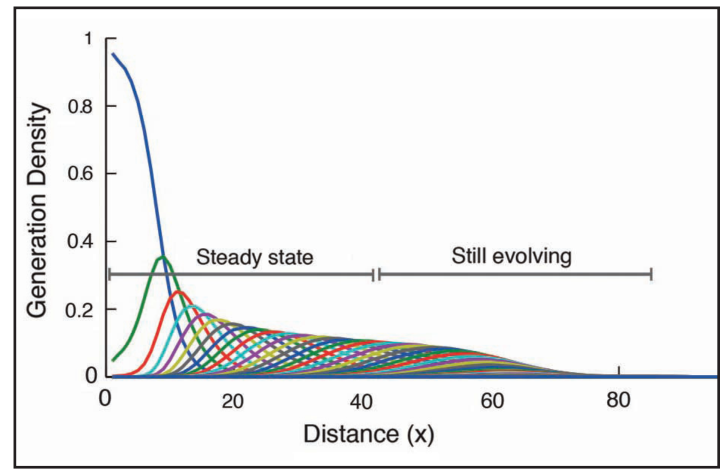


**Figure 3.** *Spatial distribution of PDE solutions for the generations* $n_i(x,t)(i = 0,1,2...)$. *The generation number* $i$ *increases from left to right, each marked with a different color. The earlier generations, which correspond to those well behind the wavefront, reach a steady state. The later generations, at the wavefront, continue to evolve for some further time. Image credit: B.L. Cheeseman.*

For each branching process, the simulation runs until all the trees terminate. We use a Lorenz curve [3], which commonly measures inequality of wealth distribution in econometrics. For example, a plot of the cumulative proportion of wealth versus the cumulative proportion of the U.S. population gives a curve that is far from a 45-degree straight line. In our cellular context, we look at the number of initial cells and ask how much their Galton-Watson-generated lineages contribute to the total final number of cells. It is highly unequal, and it grows more unequal as the cell proliferation rate increases. The progeny of a few superstars dominate the final population.

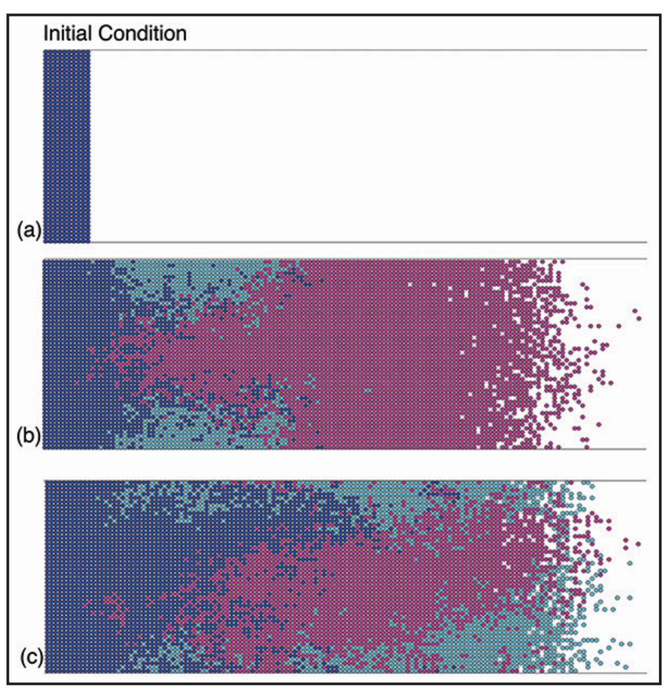


**Figure 2.** *Invasion wave and spatial distribution of agent tracings.* ***(a)*** *depicts the initial condition with 500 agents.* ***(b)*** *and* ***(c)*** *show two realizations of the travelling wave that moves progressively to the right, illustrating the largest and second largest single agent lineage tracing (pink and turquoise respectively) and the 498 other agent lineage tracings (all collected together in blue). In* ***(b)*** *there are significant differences in the agent numbers between the two largest tracings, while in* ***(c)*** *the two largest tracings have a similar number of agents. Image credit: B.L. Cheeseman.*

This method, using solutions to PDEs, has provided a genealogy with highly asymmetric lineages. It correctly identifies the existence of superstars and associated properties. These results compare very well with agent-based lineages.

Within an embryo, the gut tissue is growing everywhere during the development of the ENS. Adding domain growth to our PDE models generalizes all the methods nicely. Furthermore, this technique for determining individual data works for other PDEs that describe motility and proliferation events.

There is much interest in clonal advantage through mutation in the cancer field, from the viewpoint of 'expansion of the fittest.' We have demonstrated differential clonal expansion in an invading cell population through PDEs, agent-based models, and experiments, and argue that luck (expansion of the luckiest) may have a surprisingly large effect on differential clonal expansion.

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# The Future of High Performance Scientific Computing is Anything but Clear

*By Bruce Hendrickson and Sivan Toledo*

Cutting-edge scientific computing has relied for decades on what seems to be a never-ending parade of faster and faster computers. The continuously-growing power of supercomputers has been well documented by the TOP500 List,[1] a website that publishes a biannual list of the 500 most powerful computers in the world. However, more recent signs indicate that the underlying dynamics driving this growth are slowing down and will eventually stop, at least in their current form, in about a decade.

To help the high performance scientific community prepare for the future, three experts shared their views on the future of high performance computing with attendees at the SIAM Conference on Parallel Processing for Scientific Computing, which was held in Paris, France, this April.

Panelists Horst Simon, deputy director of Lawrence Berkeley National Laboratory (and one of the authors of the TOP500 List); Thomas Sterling, a professor and chief scientist of the Center for Research on Extreme-Scale Technologies at Indiana University; and Mike Heroux, a senior scientist at Sandia National Labs, agreed that significant improvements in supercomputing will require overcoming major challenges. But interestingly, they had completely different views on what challenges are most important.

Discussing the underlying technologies used to build processors, memories, and communication channels, Simon argued that it is hard to know what's coming next. All we know for sure is that progress is slowing down, and that the rate of progress will halve from its current level in a decade or less. He illustrated a clear slowing down in some stable metrics of progress in the TOP500 List. The rate of progress in performance of the trailing computer on the list (number 500) dropped in 2008, while the rate of progress in cumulative performance of the computers on the list dropped in 2013 (see Figure 1, on page 6). This is clearly related to the nearing end of progress in photolithography, the technology that is used to mass-produce computer chips. This technology has been advancing at a fairly constant *exponential* rate known as Moore's law from the early 1970s, when Intel produced its first processor using photolithography with a feature size of 10μm. Photolithography is still used to produce computer chips today, but the feature size dropped to around 15nm, allowing semiconductor manufacturers to cram about a million times more transistors into a chip than 45 years ago. Experts expect this process to continue for a while, down to 10nm and then 7 or even 5nm, but it is unlikely that it would go much further. At 5nm, transistors are about 20 silicon atoms wide; below that, quantum effects take over the electronic principles that currently make computers tick. Furthermore, as feature sizes shrink, the cost to develop and build chip-manufacturing plants skyrockets, making investments in them difficult to justify. At the current slower rate of progress, we will reach the 7nm or 5nm limit in about a decade.

Simon and other experts believe that in the next few years, progress will rely on two emerging technologies that still lie within the domain of conventional photolithography, which produces logic gates known as complimentary metal-oxide semiconductor (CMOS) gates. One is the three-dimensional stacking of chips, which can increase the density of memory and logic gates per unit volume. The other is silicon photonics, which can reduce power consumption and improve data-transfer performance by tightly integrating digital electronics with optical communication links. Further in the future, Simon envisions three classes of technologies that may enable additional improvements in computing power. One is the so-called post-CMOS transistor, which aims to build memory cells and logic gates that exploit rather than suffer from quantum effects. There are many candidate technologies, but it is unclear which of them can be mass produced in a cost-effective manner. The second technology is quantum computing, though it is hard to tell how significant and general-purpose it may become. The third is an early-stage technology, called neuromorphic computers, which aims to mimic brains. However, it is fairly clear that no technology is poised to take over smoothly from von Neumann architectures and CMOS devices. We may well experience a period with very little progress in hardware capabilities.

Acknowledging the possible end of CMOS scaling, Thomas Sterling made a case for radically rethinking computer architectures and programming models. He reasoned that the architecture of super-

*See* **Scientific Computing** *on page 6*

1 http://top500.org

# Math Models Examine the Effectiveness of Car-Sharing

## *Texas Students Win Top Prize for Modeling Alternative Methods of Transport*

When mass-produced cars hit the market in the early 20th century, car ownership in developed countries soared. Now, as drivers increasingly feel the economic burden of buying and owning cars, the market is undergoing a different kind of shift. Car-sharing, a form of short-term car rental frequently used for commuting, is experiencing unprecedented popularity; it offers consumers environmentally-friendly transportation without the complications and expenses of car ownership and maintenance.

But what factors determine the possible success of car-sharing in a given city, particularly from the perspective of auto and auto-sharing companies? Car-sharing's worth—for both customers and auto corporations—is contingent upon the amount of time an individual spends behind the wheel and the daily mileage. Options for car-sharing depend on customer need, and range from roundtrip and one-way sharing to multiple ownership of a single car.

The various factors that influence car-sharing's success made it a perfect topic for the eleventh annual Moody's Mega Math ($M^3$) Challenge,[1] an applied mathematics contest in which participating teams of high school juniors and seniors across the United States address a realistic problem in 14 hours using math. Sponsored by The Moody's Foundation and organized by SIAM, the Internet-based Challenge invites the top six teams to present their work to a panel of judges and compete for scholarships at Moody's Corporation headquarters in New York City.

"Moody's Mega Math Challenge provides students with an outlet to investigate real-world problems while doing mathematics they understand, and that they already have in their toolbox," said judge Ben Galluzzo (Shippensburg University), who served as a panelist at this year's final presentations, which took place on April 25. "And that's what really makes me excited about the contest, that you have this opportunity to take part in something that's different from the classroom. It gives you a taste of what real-world math would be like."

Galluzzo and fellow judges Katie Fowler (Clarkson University) and Karen Bliss (Virginia Military Institute), who was also on the panel, wrote this year's car-sharing problem.[2] The problem asked participants to classify U.S. drivers based on their extent of car usage and create a model to determine which of four given car-sharing options—roundtrip sharing, one-way sharing with manual car repositioning, one-way sharing with designated stations, and multiple ownership of cars—would work best in different locations. The problem also asked students to consider the impact of future advances, such as self-driving and alternative-energy vehicles, on car-sharing.

The $M^3$ Challenge winning team from St. John's School in Houston, Texas, strikes a pose. From left: Dwight Raulston (coach), Eric Gao, Margaret Trautner, Nancy Cheng, Daniel Shebib, and Anirudh Suresh. Photo credit: Brad Hamilton.

Mathematical models, which use various mathematical techniques to measure real-world situations and relay relevant conclusions, can identify the types of car-sharing

*See* **Car-Sharing** *on page 7*

[1] https://m3challenge.siam.org/

[2] https://m3challenge.siam.org/archives/2016/problem

# New York City Students Get a Glimpse of Math in Action

The collective "whoa" from everyone in the room said it all. With his coin "magic" trick, Tim Chartier not only captured the attention of the 50 high schoolers gathered at Manhattan's High School of Economics and Finance early in the morning on their first day of spring break, but also indelibly linked the power of math to activities they had so far never associated with the subject. Not to mention, earned them bragging rights among friends for a long time to come.

Chartier, a mathematics professor at Davidson College, proceeded to explain the "magic," a trick built upon a method of counting coins in an unusual way. "Math makes the magical logical," he said. Describing the mathematical curve that forms when launching a bird on Angry Birds, the mathematical underpinnings of movie special effects, and the math behind improving a sports team's performance, Chartier spoke the teenagers' language while simultaneously opening their eyes to a world of possibilities with mathematics.

"Sometimes kids can be good at math but they're not always sure why they're learning it, so there can be a motivational gap between math and their actual learning," said Chartier. "One of the things we are doing today is helping kids see why they should be learning math."

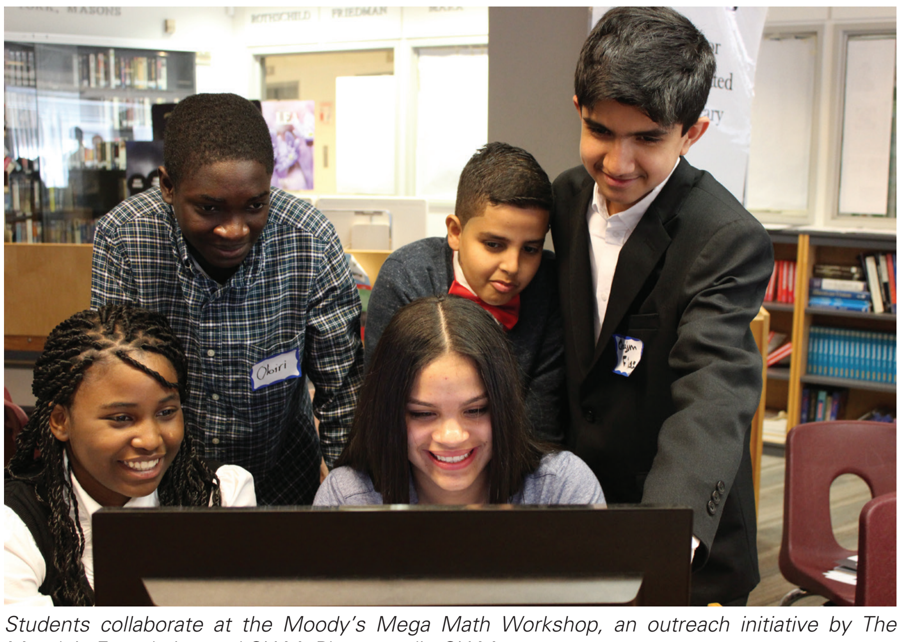
After hearing all about mathematical applications in everyday problems, the students were ready to put this information into action. Katie Fowler of Clarkson University led a mathematical modeling workshop[1] with a real-world problem: how many cats would result from the breeding of two cats over several years?

Presenting vastly different answers to this question from various animal humane societies, Fowler illustrated the need for help in the area, reinforcing the open-ended nature of mathematical modeling. "The problem of trying to understand the stray cat population can be approached on many different levels," Fowler explained. "Different groups of students will come up with very different numbers, but they'll be able to justify that based on the assumptions they make. They'll see that there is more than one way to get an answer to a problem."

Fowler explained the various components of the modeling process to the group: defining the problem statement, making assumptions, assigning variables, creating a model, and validating the model with real data. She then asked students to consider the multiple factors that could influence cat populations. Participants came up with relevant considerations, including the number of male versus female cats, mating age and times, litter sizes, the length of pregnancies, and so on. After a preliminary discussion, students split in groups to work on their math models.

[1] https://m3challenge.siam.org/newsroom/nyc-high-school-students-equate-math-real-world-solutions

Students at the Moody's Mega Math Workshop learn about math modeling and apply their knowledge to a real-world problem. Photo credit: SIAM.



Students collaborate at the Moody's Mega Math Workshop, an outreach initiative by The Moody's Foundation and SIAM. Photo credit: SIAM.

The objective of the Moody's Mega Math Workshop, an outreach initiative by The Moody's Foundation and SIAM, was to help students connect math to real-world issues. "I really enjoyed the workshop because it showed me how math is a part of our everyday lives, which you don't get to see a lot," said Destiny Santos-Ferrer of New Design High School. "We usually don't see how math connects to the real world and how it's used in multiple jobs and careers. [This workshop] incorporated what we are learning in different ways and made us more appreciative of math."

The final speaker of the day was Lindsay Hall, a software engineer at Google. After sharing her personal experiences with math and computer science in high school, Hall described the many ways she uses math at Google. She gave students a chance to ask her questions about Google products of interest to them, and proceeded to explain the underlying math in each – such as finding directions using Google Maps as a graph problem where the map calculates the shortest way to get from one "node" to another.

The workshop concluded with awards of $1,000 each in college scholarships to three random winners. The recipients, Naiomy Rangel of the High School of Economics and Finance, Maurice Avery Jr. of Marta Valle High School, and Chelsea Vicente of Queens Vocational and Technical High School, were also recognized at the Moody's Mega Math Challenge awards reception at Moody's Corporation headquarters and treated to a trip to the New York Stock Exchange to witness the Closing Bell ceremony.

"There's not nearly enough extracurricular mathematics in New York City, so we have a number of students who are always hungry for more math and some kind of challenge," said Dr. Philip Dituri, who accompanied his students to the workshop from New Design High School. "A lot of times students do modeling without being completely aware that they are modeling. Every time a kid answers a word problem in an algebra class, they model a real life situation with algebra, but they are not entirely aware that it is a modeling activity. I think this workshop gave them a real-world perspective."

*— Karthika Swamy Cohen*

# Euler in the Age of Enlightenment

**Leonhard Euler: Mathematical Genius in the Enlightenment.** *By Ronald S. Calinger, Princeton University Press, Princeton, NJ, 2016, 696 pages, $55.00.*

BOOK REVIEW
*By James Case*

As a graduate student in the history of science at the University of Chicago during the 1960s, Ronald Calinger considered writing his doctoral thesis on the life of Leonhard Euler. However, he bowed to advice from Saunders Mac Lane that the time was not yet right for such a project, since a massive effort to catalogue and translate Euler's complete works was just getting underway. Only as that poject was nearing completion did he resume work on the book he had always intended to write.

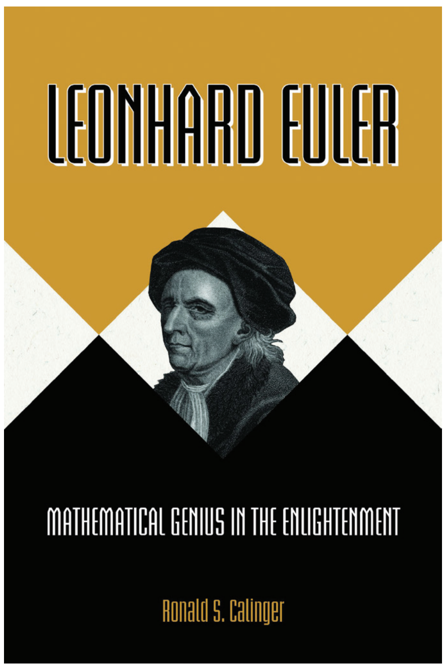
Calinger's new book is neither a mathematical nor scientific biography. As the subtitle suggests, it is largely an attempt to position Euler within the pantheon of Enlightenment figures, including thinkers such as Voltaire, Rousseau, and Montesquieu, as well as rulers like Charles II of England, Louis XIV of France, and Frederick the Great of Prussia, who chartered the royal scientific institutions that eclipsed the research performance of traditional universities for a time. There were about seventy such institutions by 1789.

The Enlightenment seems to have begun as an extension of the Copernican Revolution, during which educated Europeans discarded traditional ideas about the natural world in favor of reason and experience. Shortly thereafter, they began applying similar thinking to social questions, including the rightful purpose and proper form of both government and religion.

Not very much is known of Leonhard Euler's early life. His father Paul III served as pastor of Saint Martin's Church in the village of Riehen-Bettingen, some five kilometers from Basel, Switzerland. Aware of their son's extraordinary intellectual ability, Euler's parents soon sent him to live with his maternal grandmother in Basel, where he could attend the city's gymnasium. To supplement its rather meager offerings, they also arranged for him to be tutored by Johannes Burckhardt, a mathematically-inclined theologian. In 1720, at the age of twelve, Euler enrolled at the University of Basel to prepare for a career in the clergy. There he encountered Johann Bernoulli, who had single-handedly turned Basel into a leading center for mathematics. From the age of sixteen, Euler devoted the bulk of his attention to that discipline. At eighteen, he wrote a short paper on isochronal curves, which appeared a year later in Leipzig's *Acta Eruditorum*. That publication, together with the support of Bernoulli, led to an appointment at the recently-founded St. Petersburg Academy of Sciences.

In April 1727, 10 days before his 20th birthday, Euler left Basel aboard a Rhine riverboat, never to return. Arriving in St. Petersburg seven weeks later, he was greeted by a host of German speakers, among them Christian Goldbach, who was to stimulate Euler's interest in the (then-unfashionable) subject of number theory. The two would correspond for more than thirty years.

Members of the new academy were expected to produce practical as well as scholarly results. While mastering Russian, Euler performed studies of navigation and ship design, flood control, and map making. He also participated in the operation of the academy's workshop and sawmill, served as an examiner for the military cadet corps, worked in the office of weights and measures, and helped to modify the St. Petersburg tariff laws. Before long he drew up plans for a rudimentary water turbine—destined to replace overshot waterwheels as the power source of choice in 19th century mills—a single cylinder steam engine, and a propeller with which to drive a steamship.




*Leonhard Euler: Mathematical Genius in the Enlightenment. By Ronald S. Calinger. Courtesy of Princeton University Press.*

In 1733, Euler met and married Katharina, daughter of Swiss-born painter Georg Gsell. The marriage was apparently a happy one, despite the fact that only five of thirteen children survived to adulthood. The marriage lasted nearly forty years, with almost half of them spent in St. Petersburg.

Even before mounting the Prussian throne, Frederick the Great had dreamed of turning Berlin into an "Athens on the [River] Spree." Voltaire, among others, convinced him of the need for a royal academy of science, comparable to the ones in London, Paris, and St. Petersburg, and assured him that Euler's appointment to the new institution would bestow much glory on it. Although not very interested initially, Euler became receptive when a movement to purge the Russian court of foreign influences caused him to fear for his personal safety.

In 1741, he moved to Berlin. Berlin academy director Pierre-Louis Maupertuis soon became a fast friend, and helped to make Euler's early years in Berlin both happy and productive. However, following the conclusion of the Seven Years' War in 1763, during part of which Berlin was occupied by foreign troops, Frederick made it clear that Euler would not succeed Maupertuis as director. Thus, when Catherine the Great made him a princely offer to return to St. Petersburg in 1766, Euler jumped at the chance.

When Euler arrived in St. Petersburg for the first time, Newtonian theory was still encountering resistance from partisans of the older Cartesian and Leibnizian alternatives; the Paris academy was a stronghold of the former. The age-old practice of mingling science with mysticism did not die easily, in Europe or elsewhere. Newton himself studied alchemy, was never sure that his laws would suffice to explain all celestial motion, and suspected that divine intervention would occasionally be required to avoid potential collisions. Likewise Euler, when asked if a comet could ever strike Earth, replied that it was indeed possible but would never happen because any such catastrophe would violate God's biblical promise to protect mankind. On the other hand, critics of Newtonian theory objected to the occult nature of the "action at a distance" implicit in his law of gravitation.

In time, the debate surrounding Newtonian theory came to focus on five presumably-verifiable predictions, namely those concerning tidal motion, the shape of the planet Earth, the orbits of comets, planetary and lunar motion, and fluid dynamics. Euler eventually addressed all five issues, most prominently those concerning comets and planetary and lunar motion. He postponed his study of fluid dynamics for almost twenty years, possibly to avoid duplicating the efforts of his close friend Daniel Bernoulli. In time, the Paris Academy became a bastion of Newtonian theory.

Perhaps the most decisive episode in the debate surrounding Newtonian theory concerned irregularities in the orbit of the moon. At one point, mathematician Alexis Clairault suggested that a correction to Newton's inverse square law of gravitation involving an inverse fourth power was necessary to fully account for the observed anomalies. Several years passed before Euler was able to confirm to the satisfaction of his contemporaries that Newton's law suffices to explain all that could then be observed. It was by no means the most acrimonious dispute in which Euler became involved.

In 1751, Maupertuis' claim of priority for the principle of least action was disputed by mathematician Samuel König, who produced a copy of a 1707 letter from Leibniz to a colleague, in which such a principle was clearly enunciated. Rather than claiming priority for himself (as he might well have done), Euler magnanimously took the side of his friend Maupertuis and personally accused König of forgery before the Berlin Academy. That verdict remained more or less intact until, some 150 years later, additional copies of the 1707 letter surfaced in the Bernoulli archives.

Anyone seeking a concise account of Euler's mathematics, or particular parts thereof, should look elsewhere. Many excellent expositions are available, including [1, 2, and 3]. Calinger describes the age in which the man lived, along with the intellectual currents and political realities that drew his attention to particular problems. Is that not enough to ask of any book?

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*James Case writes from Baltimore, Maryland.*

## Scientific Computing

*Continued from page 4*

computers has remained stagnant, relying on interconnected von Neumann compute nodes. Sterling believes that new architectures will be necessary to increase the power of supercomputers beyond about $10^{18}$ floating-point operations per second. Even with current machines, the movement of data is the limiting factor in performance and consumes the bulk of the power. To circumvent this limitation, Sterling argued that future architectures will have to seamlessly blend computational and arithmetic units with memory in order to optimize data transfers, instead of optimizing the utilization of arithmetic units. He also believes that these future architectures will be based on data-flow models, rather than the von Neumann model. Programs will execute in a highly asynchronous and dynamic manner.

Mike Heroux asserted that any major change to computer architectures will have a dramatic impact on large simulation codes. Consequently, he feels that the biggest challenge will be developing software with the complexity and sophistication to actually benefit from future supercomputers, and that such software may not materialize without significant investments in software engineering. In the national labs, new simulation codes are being written not by single scientists, not even by teams of scientists, but by large collections of teams from different labs and universities. These teams consist of both application scientists and computer scientists. This scale of software development is necessary in order to build novel multiphysics, multiscale simulations. Such massive efforts require new software-engineering skills, training for the use of new tools, and new incentive structures from both publishers and funding agencies.

One clear message from the panel discussion was that a major change is coming to supercomputing. Many other sessions at the conference touched on the same topics, including minisymposium sessions on next-generation architectures, post-Moore era tuning, new types of accelerators, extreme-scale scientific software engineering, and more. There is a strong consensus that current approaches are running out of steam, but little clarity on what will come next. It is safe to say that the next decade in supercomputing will be marked by both uncertainty and opportunity.

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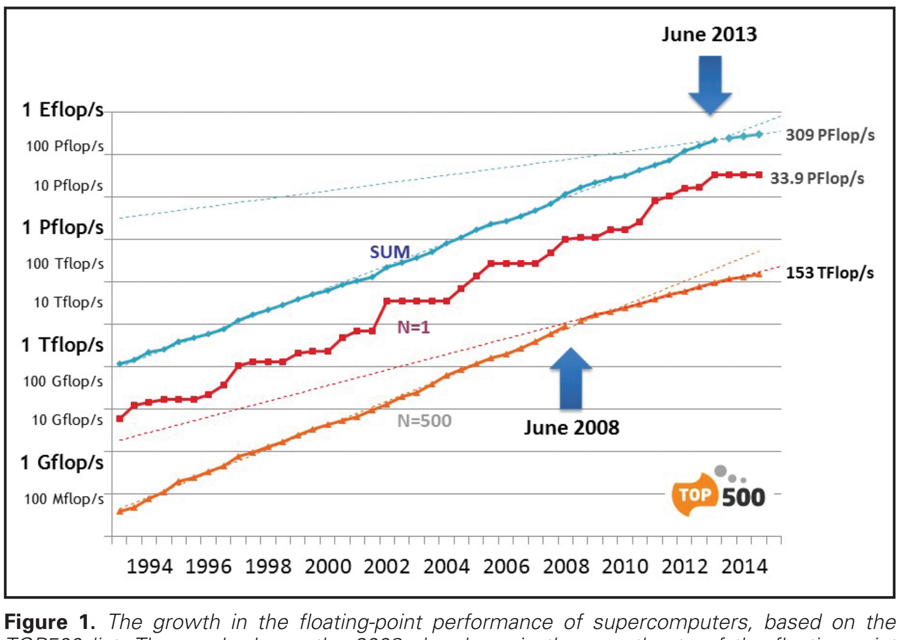


**Figure 1.** *The growth in the floating-point performance of supercomputers, based on the TOP500 list. The graph shows the 2008 slowdown in the growth-rate of the floating-point performance of the least-powerful computer among the world's 500 most powerful computers (bottom data series) and the 2013 slowdown in the growth-rate of the cumulative performance of the top 500 supercomputers (top data series). The performance of the most powerful computer (middle data series) in the world is not a smooth function of time, which makes it a poor metric for assessing growth rates. Figure courtesy of [1] © 2015 IEEE.*

*Bruce Hendrickson is the director of the Center for Computing Research at Sandia National Labs and an affiliated faculty member in the Department of Computer Science at the University of New Mexico.*

*Sivan Toledo is a professor of computer science in the Blavatnik School of Computer Science at Tel-Aviv University.*

## Car-Sharing

*Continued from page 5*

that would be most beneficial based on population density, traffic, geographical location, and demographics.

"Car-sharing is a very timely and complicated problem, in that it brings in economics as well as urban studies, and many different parts of math," said Steven Strogatz, a SIAM Fellow and speaker at the awards ceremony. "I saw students using probability theory, some geometry, some calculus, so it's reflective of the challenges of math modeling if the students do go on to professions that use math."

the expected average value for the number of miles driven per day, and integrated a weighted cumulative density function of that distribution over time. This allowed them to categorize drivers as low, medium, or high car users based on hours and daily mileage.

The team validated its model by testing in two starkly different regions—New York City and suburban Englewood Cliffs, NJ—and demonstrating that the former had a larger proportion of cars moving shorter distances while the latter had a larger proportion moving longer distances. Next, they determined which of the four given types of car-sharing would work best in four pre-determined cities.


*Members of the $M^3$ Challenge winning team from St. John's School in Houston, Texas, present their model to the judges. From left: Daniel Shebib, Eric Gao, Anirudh Suresh, Nancy Cheng, and Margaret Trautner. Photo credit: Brad Hamilton.*

To tackle the complicated Challenge question, the first-place team from Saint John's School in Houston, Texas, created a function that determines the expected number of miles driven per day based on the population density and number of driving hours for multiple regions. The students then produced a normal distribution around

"We took about 200,000 data points from a traffic survey in 2009 and found data about population density in the four cities we were assigned to analyze," said Margaret Trautner, member of Saint John's winning team. "We also used a lot of data about how humans move around in general, like how fast people walk, how much they walk per day, and how close they live to different stations."

The team determined the "price" of car-sharing for a user based on both financial and opportunity cost, or time spent by a user in combination with the value of a user's time. Charting cost versus user salary, the students factored in an individual's salary and specific situation (such as how long and how frequently he/she would need a car) to determine which car-sharing option would be most appropriate for a given user. This user-benefit model, combined with the population density of a given region, then estimated the number of potential car-share users in each region. The revenue and expenditure per user for each business model, along with the potential number of users in each region, provided an estimate of a car-sharing company's expected profit.

Government websites that analyze traffic, provide health-related information about how far people walk per day, and offer housing analysis were helpful in perfecting the model, said Trautner. Among the four cities specified by the problem, the winning team found that Richmond, VA, and Poughkeepsie, NY, would be most profitable for car-sharing, Richmond because it has more individuals overall that can afford car-sharing and Poughkeepsie because it has more individuals within a given area. The other two cities—Riverside, CA, and Knoxville, TN—wouldn't fair as well. The students determined that the best business models were the free-floating and one-way models. While the round-trip Zipcar model yields a relatively equal profit, an individual is much more likely to use a one-way car. The team then adjusted this for usage, cost, and revenue to consider the effects of alternative energy vehicles and self-driving cars.

While Trautner, who stumbled across the Challenge during an online search for scholarships, has never used car sharing herself, she was quick to note its appeal. "Cities are growing a lot, so a lot of people can't afford vehicles, but they have to get around huge cities," she said. "I'm from Houston, and it's a huge city. We don't have public transportation that works very well, so everyone has to have some sort of vehicle they can drive around in."

The champion team from Saint John's School—which will split $20,000 in scholarship money—consisted of seniors Nancy Cheng, Eric Gao, Daniel Shebib, and Anirudh Suresh, in addition to Trautner. The team, selected from over 1,100 teams across the country, was deservedly elated.

"We were looking at a picture of last year's finalists winning the championship, and it feels kind of reminiscent of that situation. Being in their shoes is a really incredible thing," said Anirudh Suresh. "It's also really cool that we're from Texas, since the other top teams are from the east coast area. To see a team from this region opens up the possibility for future years."

This was both the first team from Saint John's to ever compete in the $M^3$ Challenge and the first team from Texas to win the competition.

Dwight Raulston, coach of the champion team and instructor of mathematics and English, praised the Challenge's creativity and positive impact on students. "You're taking ideas apart and putting them together in different ways, providing your own unique contribution, and you get something out of it," he said. "I think this is, in a sense, an artistic endeavor. You're creating something for other people to use and enjoy, analogously to how artists create art."

*Visit the $M^3$ Challenge playlist on SIAM's YouTube channel*[3] *for contest videos.*

*— Lina Sorg and Karthika Swamy Cohen*

[3] https://www.youtube.com/c/siamconnect

# Particles with Negative Mass and the Krein–Moser Theory

MATHEMATICAL CURIOSITIES
*By Mark Levi*

Imagining objects with negative mass may seem like a scholastic exercise. But as it turns out, we can interpret some real physical phenomena as the occurrence of particles with negative mass.

As a glimpse of the strange world with negative masses, Figure 1 shows two masses of equal magnitude but opposite sign connected by a Hookean spring. Initially all is at rest and the spring is stretched, pulling the masses towards each other. In response, the positive mass will accelerate to the right as expected; the negative mass, being pulled to the left, will accelerate against the pull, i.e. to the right as well. So the whole system will accelerate to the right, with the distance between the masses remaining constant. Formally, the positions $x$ and $y$ of our particles satisfy

$$m\ddot{x} = F, \ -m\ddot{y} = -F,$$

where $F$ is the force of the spring. Addition gives $\frac{d^2}{dt^2}(x-y)=0,$ so that the distance $x-y=$constant (since $\dot{x}=\dot{y}$ at $t=0$) as claimed, and subtraction results in

$$\frac{d^2}{dt^2}(x+y)/2 = F/m,$$

showing that indeed the midpoint accelerates at a constant rate $F/m$.

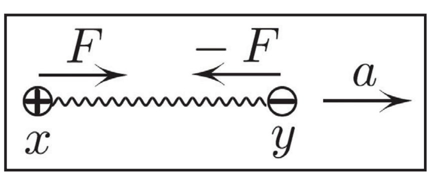


**Figure 1.** *Two masses of opposite sign connected by a spring. Acceleration happens with no external force applied.*

This acceleration occurs with no external forces applied and does not contradict Newton's second law, since the total mass of the system is zero. The fantastic world of negative masses can have spaceships which require no fuel and no external sources of energy to accelerate in any desired direction (here the discussion is limited to the one-dimensional world, but one can imagine a 3D construction along the lines of Figure 1, with the astronaut pushing or pulling on the right mass and controlling the direction of acceleration.) There is no contradiction with the conservation of energy since the kinetic energy of our system remains zero and the potential energy of the spring remains constant at all times (with the initial conditions as specified).

Consider now the regular harmonic oscillator $\ddot{x}+x=0$, with mass $m=1$ and Hooke's constant $k=1$. Multiplying both sides by $-1$ yields a mathematically equivalent system $-\ddot{x}-x=0$. Physically, we can interpret this as describing the motion of a particle of negative mass attached to a spring with the negative Hooke's constant. Until such a system is touched, it will behave as a normal mass-spring one. But when connected to a "normal" system—say, by a weak Hookean spring—an instability may result. Informally, we can trace the mechanism of this instability to Figure 1; if the positive mass "trails" the negative one, then the interaction will cause *both* masses to accelerate. This is not difficult to see formally on a simple model of two harmonic oscillators connected by a spring with a small Hooke's constant $\varepsilon$:

$$\begin{cases} \ddot{x}+x=\varepsilon(y-x), \\ -\ddot{y}-y=\varepsilon(x-y). \end{cases}$$

We have $\frac{d^2}{dt^2}(x+y)+(x+y)=2\varepsilon(y-x),$ so that $x+y$ behaves as a forced harmonic oscillator. And the forcing $2\varepsilon(y-x)$ satisfies $\frac{d^2}{dt^2}(y-x)+(y-x)=0,$ the equation of the harmonic oscillator with the *frequency* and therefore in resonance with $x+y$, causing the amplitude of oscillations of $x+y$ to grow linearly.

All this is the tip of a very nice theory developed by Mark Krein and independently by Jürgen Moser. In the 1950s, physicists working at Brookhaven National Labs made a puzzling experimental observation: a simple resonance consisting of two frequencies becoming equal led to an instability in some settings but not in others. Upon hearing of this phenomenon, Moser provided an explanation [3], and it turned out that Krein had explained the same phenomenon a few years earlier [1]. The explanation boils down to a beautiful analysis of symplectic matrices; the details can be found in the cited papers of Krein and Moser or in [2].

To give the flavor of the Krein-Moser result, we recall that the spectrum of a symplectic matrix is symmetric with respect to the unit circle, as illustrated in Figure 2 (this fact is known as the Poincaré-Lyapunov theorem). And thus a simple eigenvalue cannot leave the unit circle under small perturbation of a matrix (within the symplectic class); otherwise, an extra mirror image eigenvalue would appear. A simple eigenvalue can only leave a circle if it meets another eigenvalue. But not every meeting of eigenvalues causes them to leave the unit circle.

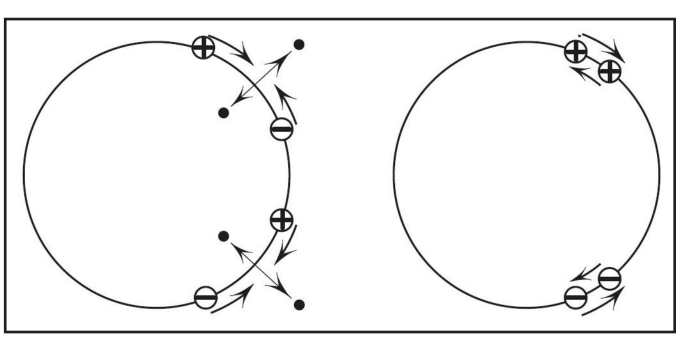

**Figure 3.** *Collision of the same-sign eigenvalues does not lead to instability; collision of the opposite-sign eigenvalues does.*

In the Krein-Moser theory, every eigenvalue is assigned a symbol + or –.[1] The "direction of rotation" of the eigenspace, as measured by the symplectic 2-form, gives the sign. And the beautiful result is that if two same-sign eigenvalues on the unit circle meet (under the deformation of a symplectic matrix), they harmlessly "pass through each other,"

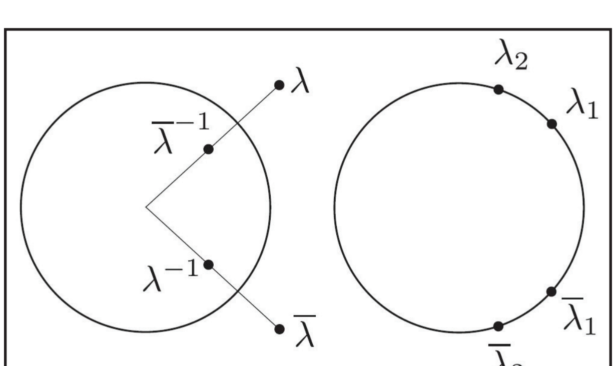


**Figure 2.** *The Lyapunov-Poincaré theorem: the symmetric spectrum of a symplectic matrix.*

*See* **Krein-Moser** *on page 11*

[1] One need not limit the attention to simple eigenvalues, but I want to skip such details.

# The Final Touch?
## *Solving Smale's 17th Challenge Problem*

*By James Case and Michael Shub*

In 1998, Stephen Smale published a list of 18 challenge problems for the 21st century. Two of them also appeared on David Hilbert's 1900 list of 23 challenge problems for the 20th century. Others were unmistakably of 20th century origin, including #17, which seeks an algorithm capable of computing approximately—in low-order polynomial average time—a single zero of a system of $n$ polynomial equations in $n$ complex unknowns.

In the late 1970s, seeing the need for a complexity theory applicable to problems in numerical analysis, Smale determined that the most useful methods would yield only approximate solutions and be prone to occasional failure. So he looked for algorithms $\alpha$ capable of providing—with high probability—an approximate solution of a particular problem $P$ of a given type in an acceptable amount of time. Specifically, if $T(P,\alpha)$ denotes the amount of time required by algorithm $\alpha$ to find an approximate solution for a particular $P$ in the class of interest, if it is possible to impose a manifold structure on that class, and if a probability measure is defined on the resulting manifold $M$, Smale asked if the "average computation time" $E(T(P,\alpha))$ is bounded by a low-order polynomial in $|P|$, the "size" of the problem $P \in M$, and the reciprocal $r$ of the probability of algorithm failure.

In 1980, Smale wrote a paper he called "The Fundamental Theorem of Algebra," in which he sought something like a constructive proof of the classical result. To that end, he defined, for any complex polynomial $f$ of order $d$, the problem

$$(P) \qquad \text{Find } \zeta \text{ such that } f(\zeta)=0.$$

The class of all such $f$ forms a manifold $M$, indeed a vector space, in an obvious fashion. He next defined $z$ to be an approximate solution of $P$ if $z$ is close enough to an actual solution $\zeta$ for Newton's method to converge rapidly (i.e. quadratically) to $\zeta$ when starting from $z$. Each actual solution $\zeta$ of $P$ is thus surrounded by a neighborhood composed of approximate solutions, which typically fill only a small portion of the basin of attraction in which $\zeta$ lies.

Basins of attraction are topologically complicated, even for a single polynomial like $z^4=1$, and need not fill the entire domain of definition. The basins themselves are typically open sets, separated by highly irregular (fractal) "curves" of measure zero, consisting of starting points from which Newton's method fails to converge. Moreover, because Newton's method can admit attracting periodic points, there may on occasion be open sets of starting points from which the method fails to converge.

Smale employed homotopy methods for finding approximate roots of univariate polynomials. Such methods work by embedding a given problem $P \in M$ in a continuum of problems $P_t \in M$, defined for $0 \le t \le 1$, of the form

$$(P_t) \qquad \text{Find } \zeta(t) \text{ such that } F(\zeta(t),t)=0,$$

where $F(z,0)=g(z)$ has an obvious root $\zeta(0)$ and $F(z,1)=f(z)$. The methods then choose a partition $\pi=\{0=t_0<t_1<\cdots<t_m=1\}$ of the unit interval and attempt to solve the problems $P_{t_i}; i=1(1)m$ successively, using $\zeta(t_{i-1})$ as a starting point in the search for $\zeta(t_i)$. The combination $F(z,t)=t\,f(z)+(1-t)g(z)$ seems the most obvious choice for $F(\cdot,\cdot)$, and is doubtless the one most frequently chosen. Indeed, Smale makes this choice when developing an upper bound on the number of intermediate problems $P_{t_i}$ to be solved approximately while solving $P_1$. Each solution curve $\zeta(t); 0 \le t \le 1$ of $P_t$ is surrounded by a (variable-width) strip of approximate solutions. There are at most $d$ such strips, which can merge and/or bifurcate repeatedly as $t$ advances from 0 to 1.

Smale subsequently investigated the computational complexity of various other problems, including the average computation time of the simplex method of linear programming. His results were later improved by Nimrod Megiddo [6]. He also sought, in collaboration with Michael Shub, to extend his results on the fundamental theorem of algebra to systems of complex polynomials.

Let $f=(f_1,\ldots,f_n)$ be a system of $n$ real or complex polynomials in the $n$ unknowns $z_1,\ldots,z_n$ and consider the class of problems

$$(P') \qquad \text{Find } \zeta=(\zeta_1,\ldots,\zeta_n) \text{ such that } f(\zeta)=0.$$

It is easy enough to impose a manifold structure on the set of all such systems $f$, and to define a probability measure on the resulting $M$.

In order to take advantage of *Bezout's* theorem on the number of zeros of a system of complex homogeneous polynomials, they began by letting $d_1,\ldots,d_n$ be the degrees of $f_1,\ldots,f_n$, and homogenizing each $f_i$ by multiplying each constituent monomial by the power of an auxiliary variable $z_0$ needed to increase its degree to $d_i$. The result is a system of $n$ homogeneous polynomials of degrees $d=(d_1,\ldots,d_n)$ in the $n+1$ variables $z_1,\ldots,z_n$. A variant of Newton's method, known as the projective Newton's method, applies to such systems.

If $\mathcal{H}(d)$ is the vector space of all homogenous systems $f=0$ of degree $d$, then $\dim\mathcal{H}(d)$ can be shown to be $N=\Sigma_{i=1}^{n} C_n^{n+d_i}$ where $C_n^m$ is a binomial coefficient, and the number of solutions $\zeta=(\zeta_1,\ldots,\zeta_n)$ is known by Bezout's theorem to be $B=\Pi_i d_i$. In particular, if each $d_i$ is 2, then $N \sim n^3/2$ while $B=2^n$. The number of solutions $\zeta$ is thus exponential in $n$, dashing any hope for a polynomial-time algorithm that produces all solutions.

With only a few equations of high degree, the problem $P'$ can be reduced via exact symbolic techniques (Gröbner bases, resultants, and the like) to that of solving a univariate polynomial of degree $B$ in polynomial time. But that fact alone does not lead to a solution of Smale's problem #17, even in combination with the aforementioned homotopy methods.

In a series of five papers co-authored with Shub—which came to be known as the Bezout series—Smale established the surprising fact [7] that a zero of $n$ complex polynomial equations in $n$ complex unknowns can be found approximately, on average, in time polynomial in $N$ alone. But the proof demonstrated only existence. It was neither constructive nor uniform.

In 2008-2009, a pair of papers by Carlos Beltrán and Luis Miguel Pardo [1, 2] described a probabilistic algorithm—where the machine must first select at random a point of departure from which to launch the chosen algorithm $\alpha$—for solving $P'$ *with high probability* in polynomial average time. This led some to consider #17 solved. Others disagreed, on the ground that Smale was seeking a deterministic method that would work for certain.

Beltrán and Pardo select at random an $n \times (n+1)$ matrix $A$ and solve $A\zeta=0$ for $\zeta \neq 0$. They then select, again at random, a homogeneous system $g$ of higher degree that vanishes at $\zeta$. The resulting pair $(g,\zeta)$ then serves as a starting point for the required homotopy, and the computation time $T(P',\alpha)$ is averaged over the two random choices.

In 2011, Felipe Cucker and Peter Bürgisser announced a deterministic variant of the Beltrán-Pardo algorithm for solving $P'$, applicable only to a restricted range of degrees and dimensions [4]. And finally, in July of 2015, Pierre Lairez revealed a complete solution [5] of Smale's problem.

Instead of choosing $A$ and $g$ at random, Lairez uses the input $f$ itself as a random element. He chooses a precision $\varepsilon$, and rounds off the coefficients of the polynomial system $f$ to that precision, thereby producing an approximation $g$ of $f$. Lairez then fixes $g$ and considers the set of all $f$ that round to the same $g$. The coefficients of $h=f-g$ are then effectively random, and may be used to produce a random matrix $A$ and a random non-linear system $\bar{g}$ vanishing at a non-zero solution $\zeta$ of $A\zeta=0$. By doing so, Lairez reduces the problem $P'$ to a member of the class shown by Beltrán and Pardo to be solvable in average polynomial time.

Lairez spoke about all this at the Simons Institute in Berkeley, CA, last December. He ended his talk by posing a problem 17*bis* asking for proof that $P'$ can be solved approximately by an algorithm $\alpha$ whose average running time $E(T(P',\alpha))$ is a low-order polynomial in the two most natural measures of the size and tractability of a given $P' \in M$, namely the length of the input sequence and the condition number of the derivative matrix $\nabla f$ at a root. Such an estimate has already been developed [3] under the hypothesis that geodesics in $M$ can be adequately approximated with respect to a metric that arises naturally in the study of homotopy methods.

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*James Case writes from Baltimore, Maryland. Michael Shub works in dynamical systems and the theory of complexity of real number algorithms. Steve Smale was his 1967 Ph.D. thesis advisor.*

# Network Data: Dealing with Incompleteness and Bias

*By Sucheta Soundarajan and Jeremy D. Wendt*

While network analysis is applied in a broad variety of scientific fields (such as physics, computer science, biology, and the social sciences), the methods used to construct networks and the resulting bias and incompleteness have drawn more limited attention. For example, in biology, gene networks are typically developed via experiment – many actual interactions are likely yet to be discovered. In addition to this incompleteness, the data-collection processes can introduce significant bias into the observed network datasets [2, 3]. For instance, a classic random walk used to observe part of the World Wide Web network would be more likely to find high-degree nodes than a random selection of nodes [1]. Unfortunately, such incomplete and biased data collection methods are often necessary.

At the recent Workshop on Incomplete Network Data (WIND),[1] held at Sandia National Laboratories[2] in Livermore, CA, researchers from academia, industry, and national labs gathered to discuss perspectives on dealing with incomplete network data. WIND was organized by Tina Eliassi-Rad (Northeastern University), James Ferry (Metron, Inc.), Ali Pinar (Sandia), and C. Seshadhri (University of California, Santa Cruz).

The workshop highlighted a host of areas with biased graph samples. Dennis Feehan (University of California, Berkeley) and Forrest Crawford (Yale University) both discussed a particularly interesting problem from the social sciences – determining how to accurately estimate the size of hidden or rare groups in massive populations by querying survey respondents. Bradley Huffaker (Center for Applied Internet Data Analysis, University of California, San Diego) presented problems related to obtaining an accurate map of the Internet, and Jaiwei Han (University of Illinois at Urbana-Champaign) described techniques for supplementing explicit graphs using unstructured text mining.

Three main approaches emerged. The first approach involves estimating properties or characteristics of the global network, given only a partial observation of that network. For example, can one estimate the number of triangles (i.e., “A” knows “B” knows “C” knows “A”) in the full network with only partial access to the full network data (e.g., Tammy Kolda, Sandia)? The second approach entails performing data collection in such a way as to reduce bias or increase the quality of the information obtained. For instance, how can one sample a node uniformly and at random from a graph, where access to data is through a random walk-like crawl (e.g., Ravi Kumar, Google)? The third approach identifies algorithm degradation resulting from noise or incomplete data and designs algorithms to be more robust. For example, local spectral methods provide results akin to full-graph spectral methods, without the effects of problems in distant parts of the graph (e.g., Michael Mahoney, University of California, Berkeley; David Gleich, Purdue University).

These categories are complementary, and a real-world application of network analysis on incomplete data would ideally incorporate all three techniques. The third category (understanding the robustness of existing algorithms against noise or incompleteness) is an important first step in any such unified approach.

David Kempe (University of Southern California) discussed the problem of algorithm robustness at the workshop. He pointed out that in real-world network data, “Noise is the norm, not the exception,” and that understanding the effects of noise on algorithmic tasks is critical. To illustrate this point, he considered the problem of influence maximization; in a network setting, if we assume that an individual’s beliefs can affect their neighbors’ beliefs, which nodes’ beliefs should we influence to have the greatest effect on the beliefs of the population as a whole? Kempe argued that the influence probabilities (i.e., the probability that node “A” will influence the belief of node “B”) can have a large effect on which nodes are selected; but any estimates of these probabilities are likely to be inaccurate!

Kempe also considered the effect of noise or incomplete data on community detection (the problem of clustering the nodes of a network into cohesive groups). He argued that the output of community detection methods can also be significantly affected by noise or missing edges in the network. For example, missing edges might lead an algorithm to identify two communities. If those edges were to be present, the algorithm would find only one community. Kempe argued that community detection on incomplete network datasets, while not sufficient for drawing conclusions, may be appropriate for suggesting hypotheses, which are then verified by other means.

Along similar lines, Anil Vullikanti (Virginia Tech) considered how noise can affect the core decomposition of a graph. A core of a graph is, in essence, a ‘dense’ or ‘central’ part of the graph and, among other applications, can be used to measure the importance or centrality of nodes in the network. Through experimental results, Vullikanti demonstrated that k-cores are unstable when the network is perturbed in degree-biased ways (that is, the probability of a perturbation affecting a node depends on the number of connections the node has). This is a critical problem because one of the most common ways of obtaining network data (crawl via breadth-first search) leads to just this kind of degree-biased sampling.

Other research presented during the workshop proposed techniques to overcome missing data or noise, including strategies for counteracting bias or generating more accurate network samples. Many attendees presented early solutions that show great promise.

However, the consensus among WIND attendees was that incomplete data presents a daunting challenge to performing accurate network analysis. Several critical questions remain: How do we measure or estimate the noise, bias, or incompleteness of network datasets? What tests could we run to thoroughly assess the effects of these data errors on later analyses?

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[1] http://eliassi.org/WIND16.html

[2] Sandia National Laboratories is a multi-program laboratory managed and operated by Sandia Corporation, a wholly owned subsidiary of Lockheed Martin Corporation, for the U.S. Department of Energy’s National Nuclear Security Administration under contract DE-AC04-94AL85000.

# Academia and Industry Meet at Emory University's SIAM Student Meeting

A major goal of SIAM student chapters is to motivate interest in applied mathematics and computational science. Chapters offer students opportunities to share ideas with peers and faculty members across a broad range of disciplines, and explore career opportunities in both industrial and academic environments.

With this in mind, the Emory University Chapter of SIAM (Atlanta, GA) hosted "Math & Industry: Happily Ever After" this past March with the dual aim of highlighting similarities and differences between industrial and academic environments and emphasizing the value of mathematics as a versatile and powerful tool for real-life applications.

Invited speaker Lior Horesh shared his experience and expertise as both a mathematician in industry and a researcher in academia. As a research scientist in the Mathematical Sciences and Analytics Department at the IBM T.J. Watson Research Center and an adjunct associate professor in the Department of Computer Science at Columbia University, Horesh straddles both worlds. As undergraduate attendee Claire Lin put it, "Dr. Horesh spoke from his own experiences in collaborative research settings, and addressed the essential concerns and challenges of working in interdisciplinary fields. For applied mathematicians nowadays, interactions between academia and industry are inevitable."

Lior Horesh speaks to attendees of Emory University's SIAM student chapter event, "Math & Industry: Happily Ever After." Photo credit: Alessandro Barone.

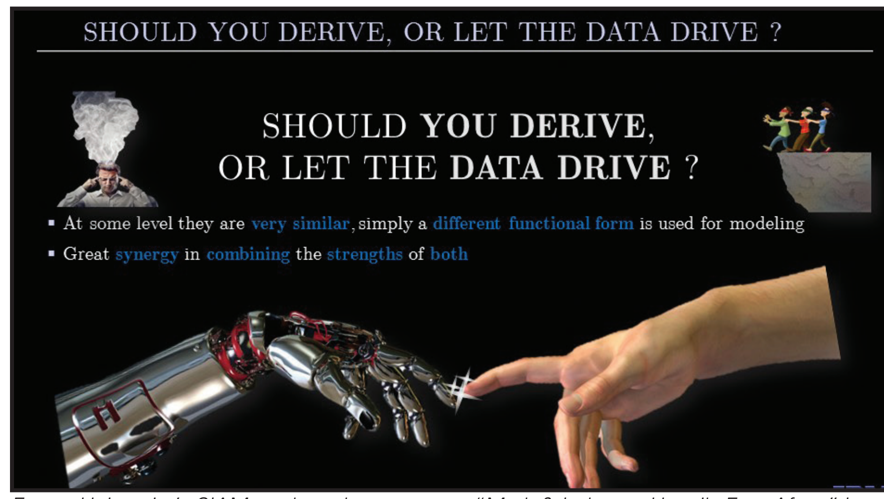
Horesh began by focusing on the changing nature of modern research, namely that a single researcher can no longer tackle many of today's challenges. In the past, examining the individual strength of a research group's members quantified its effectiveness. However, as the breadth of projects extends beyond the capabilities of any single individual, the mentality for assessing capabilities is shifting towards collective intelligence: the shared intelligence resulting from the joint efforts, cooperation, and rivalries of numerous individuals. This mindset is exemplified, for instance, by the success of Wikipedia, Linux, CAPTCHA, and other open innovation policies that harness collective knowledge to great effect. With the recent advancements and growing interest in the field of artificial intelligence (AI), the collaborative scope can be even further extended, as stated by the MIT Sloan Center for Collective Intelligence, to investigate "how people and computers can be connected so that—collectively—they act more intelligently than any person, group, or computer has ever done before."[1]

Horesh then gave examples of how both academia and industry approach this challenge. Industries like IBM provide a collaborative research environment that joins professionals from a broad range of scientific and technical fields. In such a context, the diversity of skills fosters interactions across disciplines that lead to innovative ideas. The broad scope of problems researchers face at industrial research institutions not only encourages continuous learning and professional evolution but also creates more robust, scalable research output, capable of surpassing the proof stage and delivering effective solutions. Thus, industrial research institutions offer a stimulating environment that increases the productivity of each individual and the quality of the group's collective work.

However, diverse and high-impact research cannot abstain from academic influences and external collaboration. Horesh detailed academic interactions in industrial environments that exist in the form of teaching/mentoring and research guidance. Collaboration with academic peers is encouraged and often leads to joint publications, conference co-chairing, joint patents, and internship programs. For those from academia, interacting with the industrial realm is an opportunity to explore and draw useful insights and perspectives on current work. Indeed, these interactions allow both sides to critically reconsider conventions, practices, and prejudices, and also introduce new, interesting problems and challenges to solve.

"I particularly liked the insight Lior gave about research (and its motivations) done in industry versus academia," said Samy Wu, a graduate student in the applied mathematics program at Emory. "The talk was informal enough to engage a large audience and technical enough to convey important details."

1 http://cci.mit.edu/




Emory University's SIAM student chapter event, "Math & Industry: Happily Ever After." Image credit: Lior Horesh.

Horesh concluded by describing the ideal symbiosis of industrial and academic environments. The two sides share the universality of mathematical models, which encompass a large variety of real-world problems, and the promise of fruitful collaborations should persuade the mathematics and science communities to persevere through the challenges posed by linking industry and academia. Although physicists, computer scientists, business leaders, and mathematicians all speak fundamentally different languages as a byproduct of the environments in which they learn their craft, creating effective means of communication between these disparate worlds is paramount for both understanding complex problems outside the domain of expertise and attaining solutions to pressing real-world problems.

"Dr. Horesh gave compelling arguments for the benefit of embracing work with individuals of diverse backgrounds, be it fields of study or industrial and academic affiliations," said Clarissa Garvey, a graduate student in Emory's computer science program. "I thoroughly enjoyed the presentation and ensuing discussion."

Horesh encouraged the students to value and learn from the differences while embracing the difficulties of cross-disciplinary research. And above all, to dare, explore, and try, even if we stand chances to fail. As Einstein said, "If we knew what it was we were doing, it would not be called research, would it?"

*— Sofia Guzzetti, for Emory University SIAM Student Chapter*

## Krein-Moser

*Continued from page 8*

staying on the unit circle, and thus not causing an instability. But the colliding eigenvalues of opposite sign "bump" each other off of the unit circle, as in Figure 3 (on page 8). We can interpret this collision as a same-frequency resonance between two oscillators, one of them involving a negative mass. And the Krein-Moser sign of an eigenvalue can be interpreted as the sign of an imagined mass.

*All figures in the article are provided by the author.*

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# Applied Algebra and Geometry: A SIAGA of Seven Pictures

*By Anna Seigal*

*SIAM's brand-new journal, the SIAM Journal on Applied Algebra and Geometry, will feature exceptional research on the development of algebraic, geometric, and topological methods with strong connections to applications. The cover of the new journal shows seven pictures. By describing these pictures and discussing the topics they represent, we hope to give readers a glimpse into the world of algebraic, geometric, and topological problems of interest to applied mathematicians.*

*This article is the final installment of a three-part series.*

## 5. Geometric Modeling

### The Context

Geometric modeling is an area of applied mathematics that uses piecewise polynomial functions to build computer models that describe shapes in space.

One tool that is used for such modeling is a parametric curve called a *Bézier curve*, named after the French engineer Pierre Bézier who worked in the automotive industry.

A Bézier curve models smooth motion through time or space. Each curve is defined by a number of *control points*, which specify its shape and location. These points simplify manipulation of the curve on a computer interface; changing the location of the control points causes a reliable change in the curve.

A collection of $d+1$ control points $P_0,\ldots,P_d$ defines a Bézier curve of degree $d$. The simplest example is when the degree is 1, with two control points $P_0$ and $P_1$. In this case, the Bézier curve is the line that connects the two points

$$B(t)=(1-t)P_0+tP_1$$

for $t$ between 0 and 1. A degree 2 example is given by

$$B(t)=(1-t)^2P_0+2t(1-t)P_1+t^2P_2,$$

and in general we have

$$B(t)=\sum_{i=0}^{d}\binom{d}{i}(1-t)^{d-i}t^iP_i.$$

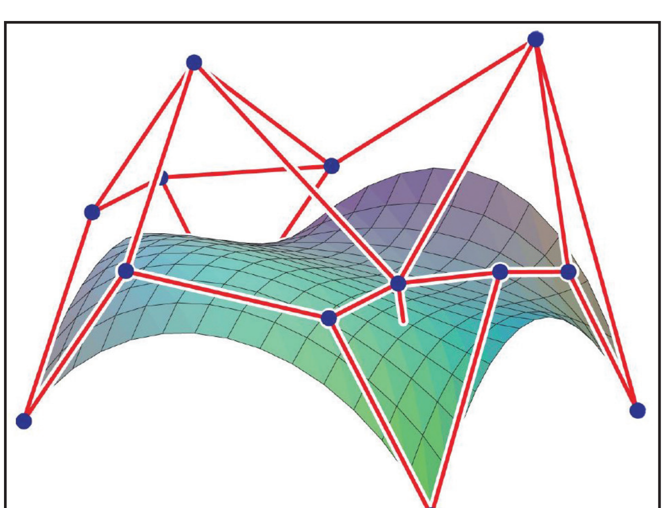


**Figure 1.** *A generalization of a Bézier curve to a two-dimensional Bézier surface. Figure courtesty of [4].*

A Bézier curve has a 'control polygon' associated with its control points, which is found by taking the line segments connecting adjacent control points. The convex hull of the control polygon contains the curve, and the control polygon has other interesting properties as well. For instance, it can be used in approximation of the original curve.

### The Figure

Figure 1 shows a generalization of Bézier curves to a two-dimensional Bézier surface. It is from [4].

Bézier surfaces provide convenient ways to make smooth two-dimensional surfaces, such as for the design of car parts. A Bézier surface is defined in terms of a collection of control points in three-dimensional space

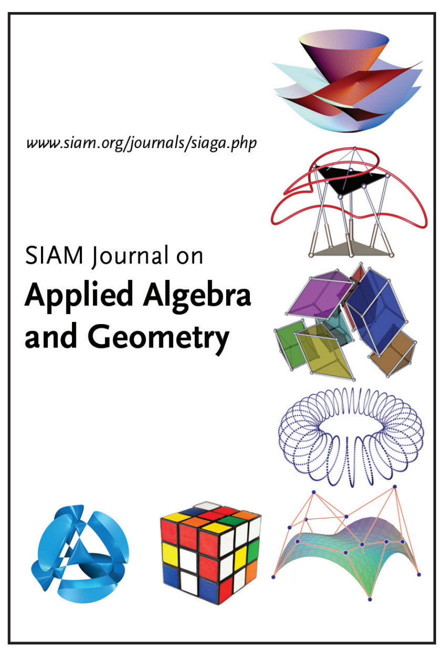
$$\{P_{0,0},\ldots,P_{d_1,d_2}\},$$

which now is indexed by two indices rather than one. It is described parametrically by

$$B(t_1,t_2)=\sum_{i_1=0}^{d_1}\sum_{i_2=0}^{d_2}\left(\binom{d_1}{i_1}(1-t_1)^{d_1-i_1}t_1^{i}\right)\times\left(\binom{d_2}{i_2}(1-t_2)^{d_2-i_2}t_2^{i_2}\right)P_{i_1,i_2}.$$

A list of $(d_1+1)(d_2+1)$ control points gives a surface of degree $d_1d_2$ via this process.

The control points are shown in blue, and the Bézier surface is shown below them in green. We now have a two-dimensional analogue of the control polygon. The surface sits below the convex hull of the control points, which is identified by red lines connecting the blue points. This polyhedral structure connects geometric modeling to polyhedral geometry, described in last month's installment.[1]




*The new SIAM Journal on Applied Algebra and Geometry (SIAGA).*

Applications often demand the investigation of further properties of Bézier curves and surfaces, such as how they intersect with one another. One step in the process is obtaining an implicit formula from a surface's parametric description. That is, finding the relations amongst the coordinates that are satisfied for all points on the surface. Here, computational algebraic geometry tools are very useful.

## 6. Tensors

### The Context

Tensors are the higher-dimensional analogues of matrices. They are like matrices, but with three or more dimensions, and are represented by an array of size $n_1\times\cdots\times n_d$, where $n_k$ is the number of 'rows' in the $k$th direction of the array. The entries of the tensor $A$ are denoted by $A_{i_1\ldots i_d}$, where $i_k\in\{1,\ldots,n_k\}$ identifies which row in the $k$th direction you are viewing. Just as for a matrix, the entries of a tensor are elements in some field, for example real or complex numbers.

Tensors occur naturally when it makes sense to organize data by more than two indices. For instance, if we have a function that depends on three or more discretized inputs $f(x,y,z)$ where $x\in\{x_1,\ldots,x_{n_1}\}$, $y\in\{y_1,\ldots,y_{n_2}\}$, and $z\in\{z_1,\ldots,z_{n_3}\}$, then we can organize the values of $A_{ijk}=f(x_i,y_j,z_k)$ into a tensor of size $n_1\times n_2\times n_3$. Tensors are increasingly widely-used in many applications. This is especially true of signal processing, where the uniqueness of a tensor's decomposition allows us to find the different signals comprising a mixture. They have also been used in machine learning, genomics, geometric complexity theory, and statistics.

[1] https://sinews.siam.org/DetailsPage/tabid/607/ArticleID/798/Applied-Algebra-and-Geometry-A-SIAGA-of-Seven-Pictures.aspx

Data analysis techniques are currently limited to a matrix-centric perspective. Tremendous effort to extend the well-understood properties of matrices to the higher-dimensional world of tensors is attempting to overcome this limitation. A greater understanding of tensors paves the way for exciting new developments that can cater to the natural structure of tensor-based data, for example, in experimental design or confounding factor analysis. Such understanding and analysis uses interesting and complicated geometry.

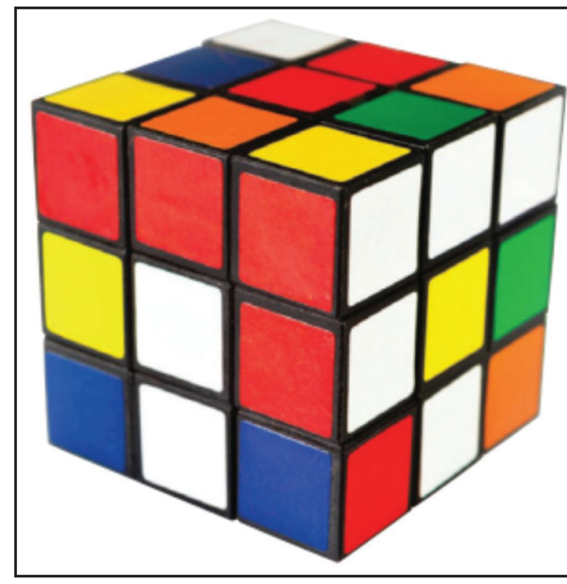


**Figure 2.** *The Rubik's Cube is also a cartoon of a tensor of size 3x3x3. Rubik's Cube® used by permission, Rubik's Brand Ltd. www.rubiks.com.*

One requirement for computability of a tensor is a good low-rank approximation. Tensors of size $n_1\times\cdots\times n_d$ have $n_1\ldots n_d$ entries, and this quickly becomes unreasonably large for applications. Matrices can be analyzed via their singular value decomposition, and the best low-rank approximation is obtainable directly from this decomposition by truncating at the $r$th largest singular value. For tensors we can also define useful related notions such as eigenvectors, singular vectors, and the higher order singular value decomposition.

### The Figure

In addition to depicting the well-known Rubik's Cube, Figure 2 is a cartoon of a tensor of size $3\times3\times3$. Such a tensor consists of 27 values.

To understand the structure contained in a tensor, we use its natural symmetry group to find a presentation that is simple and structurally transparent. This motivation also underlies the Rubik's puzzle, although the symmetries can be quite different: a change of basis transformation for the tensor and a permutation of pieces for the puzzle.

Despite being small, a $3\times3\times3$ tensor has interesting geometry. A generic tensor of size $3\times3\times3$ has seven eigenvectors in $\mathbb{P}^2$. We show in [1] that any configuration of seven eigenvectors can arise, provided no six of the seven points lie on a conic.

## 7. Visualization of Algebraic Varieties

### The Context

There is a vast mathematical toolbox of techniques that enable understanding of algebraic varieties. It's great when we can actually draw the algebraic variety in question using visualization software. When possible, this allows us to make the most direct observations.

Although it poses an obvious restriction on the number of dimensions in which we can work, even visualizing particular slices through our variety of interest is structurally revealing. Large polynomials with many terms can be hard to grasp. Modern-day computer tools convert these equations into helpful pictures, aiding comprehension.

### The Figure

Figure 3 shows a *Kummer surface*. It was made by Oliver Labs using the visualization software 'Surfex.' Many beautiful pictures have been created in this way: for more, see the picture galleries from the 'Imaginary: Open Mathematics' website.[2]

This figure is an example of an irreducible surface in three-dimensional space of degree four. In general, these surfaces have at most 16 singular points. Kummer surfaces are those that attain this upper bound. The 16 singular points represent the 2-torsion points on the Jacobian of the underlying genus 2 curve.

Figure 3 also represents the problem-solving areas of coding theory and cryptography, which contain a broad range of applied algebra and geometry. The group law on an elliptic curve is fundamental for cryptography. Similarly, the group law on the Jacobian of hyperelliptic curves has been used for cryptographic purposes (see [2, 3]). The latter is by Kristin Lauter from Microsoft Research, who is president of the Association for Women in Mathematics (AWM).

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[2] https://imaginary.org/

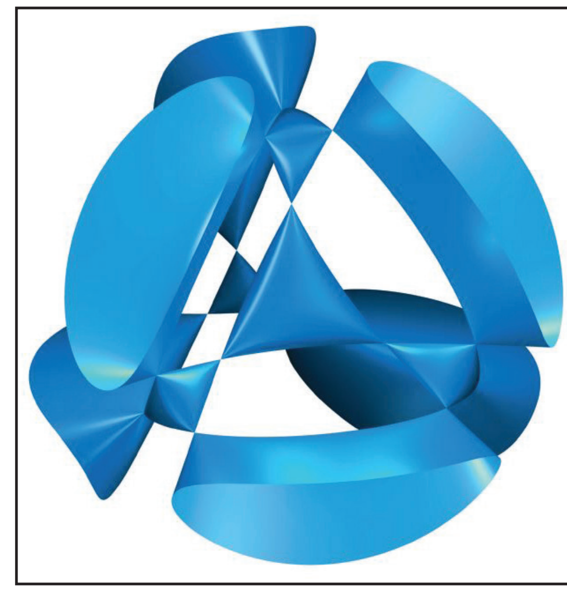

**Figure 3.** *A Kummer surface has applications in coding theory and cryptography. Image credit: Oliver Labs.*