Newsjournal of the Society for Industrial and Applied Mathematics sinews.siam.org

siam news

Volume 54/ Issue 3
April 2021

Control and Integrability in Figure Skating

By Meghan Rhodes, Vaughn Gzenda, and Vakhtang Putkaradze

Figure skating is a popular athletic endeavor, even taking the prime-time slot during televised broadcasts of major sporting events like the Winter Olympic Games. The grace, precision, and elegance of the athletes—in addition to the enchanting effects of dance, music, and costumes—make this sport a perennial favorite for many viewers.

The patterns that skaters carve into the ice have intrigued a wide audience since the 19th century [2]. As a hobby, skaters designed intricate patterns and attempted to trace those designs into the ice. This pastime soon grew in popularity; the patterns became known as "figures" and the hobby was termed "figure skating." Some of us might still remember televised broadcasts of competitions that included "compulsory figures," during which athletes could trace patterns at any speed but were only allowed a certain number of pushes off the ice. These contests are not part of official competitions anymore; they have separated into an independent branch with its own events that differ from mainstream Olympic-style figure skating.

While the acrobatic dance programs and jump-filled free skate performances of mainstream figure skating are certainly more spectacular for viewing—and more challenging athletically—the mathematical descriptions of continuous skating paths are just as interesting. Skaters control their motion by changing the direction of their skates and the position of their arms, legs, and torso, thus altering the moments of inertia and location of the center of mass. Such control is produced in the body's frame, whereas the figures are traced on ice — the coordinate frame fixed in space (spatial frame). The exact nature of the mechanism that a skater employs by mapping the body to the spatial variables—which one achieves with years of careful practice—is very complex and not well understood. This lack of understanding is evident in figure skating instructional methods. For example, each figure skating element can be executed in many ways. Different body positions are considered "optimal" depending on the execution technique of the element in question. When learning a new element, skaters often experiment with technique and try to simply remember the physical feeling of a successful attempt in order to recreate the success (rather than determining the physics behind the attempt). The level to which skaters analyze their movements and the resulting impact on the physical properties varies based on the individual, but significantly examining this relationship is not a common practice.

Though one can technically derive a full mathematical model of a skater on ice, such a model is hardly informative for theoretical understanding. The model would have to incorporate the motion of the head, torso, and every limb — all controlled simultaneously. This challenging problem is likely a good candidate for modern machine learning methods, such as reinforcement learning. However, we are interested in understanding the problem by designing the simplest realistic mechanical model as possible. We focus on the skater's motion without the blade's friction with ice, which is a reasonable assumption for a description of the motion on short to intermediate time scales.

During the continuous motion of a skater on ice, the skate in contact with the ice can only move in the direction of the blade. When the skate turns with respect to the ice, the direction of motion also changes. There is thus a constraint on the velocities of the skate on ice. One cannot write the skate velocity constraint in terms of coordinates only, meaning that the mechanical system

See **Figure Skating** *on page 4*

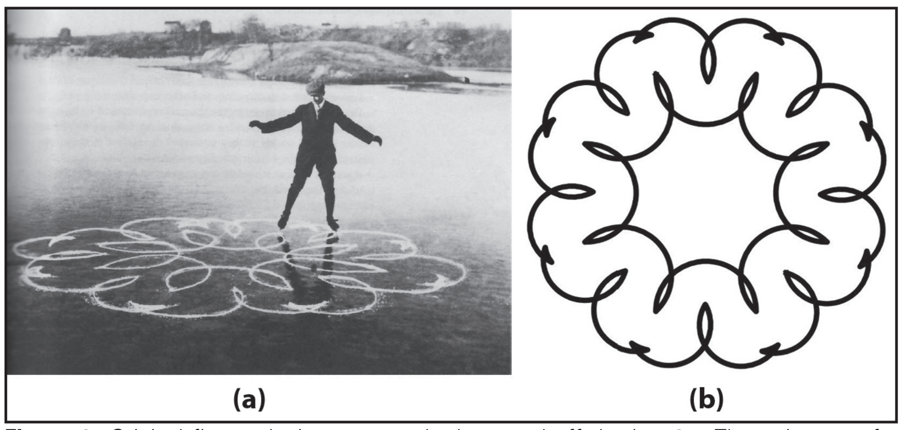

Figure 1. *Original figure skating patterns both on and off the ice.* **1a.** *The trajectory of a skater on ice from one of the original figure skating patterns.* **1b.** *The same figure, computed by combining the trajectory of a controlled Chaplygin sleigh. Figure courtesy of Meghan Rhodes and Vakhtang Putkaradze.*

A Measure of Morphodynamics

By Mattia Serra and L. Mahadevan

One of the grand challenges of modern biology is understanding the way in which a complex, multicellular organism arises from a single cell via spatiotemporal patterns that are repeatable and reproducible across the tree of life. As the organism grows, its cells change their number, size, shape, and position in response to genetic, chemical, and mechanical cues (see Figure 1a). Four-dimensional microscopy (three spatial dimensions and time) is beginning to illuminate how these cues impact the fate of cells and the geometric form of tissues and organs that constrain and enable function at multiple scales [3]. Even though individual cells might seem to move chaotically, the large-scale, collective cell movements within tissues resemble a choreographed ballet and raise a few natural questions:

(i) How can we quantify the patterns and predict the formation of different organ systems?

(ii) How can we understand these patterns from a biophysical and biochemical perspective as a function of the way in which cells divide, grow, and move in response to environmental cues?

(iii) How can we control these movements to intervene and correct pathological development or guide tissue development in situations like organoid formation?

Here we focus on the properly invariant quantification of large-scale cellular movements and draw inspiration from the study of objective transport barriers in hydrodynamics [2, 5]. Just as it is more meaningful to focus on the large-scale coherent structures in a complex flow rather than track individual particles, we believe that it is useful to quantify the large-scale motions that characterize tissue morphogenesis. Any framework that aims to analyze spatiotemporal trajectories in morphogenesis requires a self-consistent description of cell motion that is independent of the choice of reference frame or parametrization. This objective (frame-invariant) description of cell patterns ensures that the material response of a deforming continuum—e.g., a biological tissue—is independent of the observer [7].

One can quantify this idea mathematically by considering two reference frames that describe cellular flows and are related to each other via the transformation $\overline{\boldsymbol{x}}(t)=\boldsymbol{Q}(t)\boldsymbol{x}(t)+\boldsymbol{b}(t)$, where $\boldsymbol{Q}(t)$, $\boldsymbol{b}(t)$ are a time-dependent rotation matrix and translation vector respectively. A quantity is objective (frame invariant) if the corresponding descriptions in both frames transform appropriately: scalars $c(t)$ (e.g., concentration) must remain the same with $\overline{c}(t)=c(t)$; vectors $\boldsymbol{v}(t)$ (e.g., velocity field) must transform via the rule $\overline{\boldsymbol{v}}(t)=\boldsymbol{Q}(t)\boldsymbol{v}(t)$; second-order tensors $\boldsymbol{A}(t)$ (e.g., strain rate) must transform via the rule $\overline{\boldsymbol{A}}(t)=\boldsymbol{Q}(t)\boldsymbol{A}(t)\boldsymbol{Q}^T(t)$; and so forth [7]. It is thus immediately apparent that objects like velocity fields, streamlines, and vorticity—which are typical outputs from microscopy following some elementary image analyses—are frame dependent. Therefore, any metrics that are based on these objects for comparative purposes are likely erroneous, owing to their inability to remove dependence on artifacts that are associated with variations in the choice of reference frames (see Figure 1b). Indeed, most Eulerian approaches that characterize fluid or cellular flows suffer from this flaw, despite having served as workhorses for a long time [1, 5]. So, how can we do better?

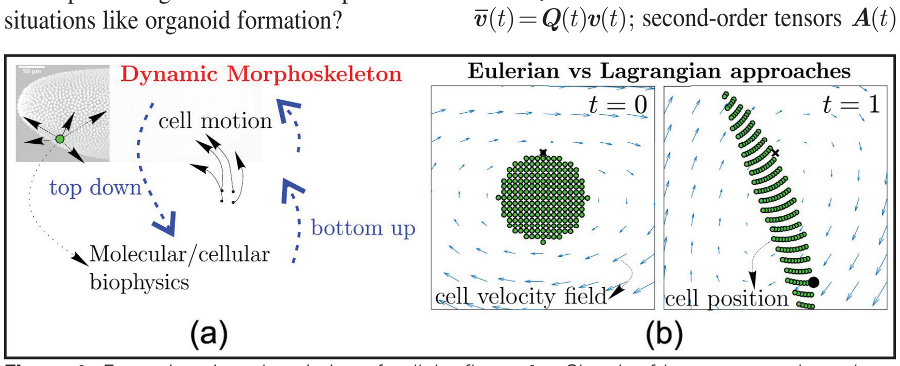

Figure 1. *Frame-invariant description of cellular flows.* **1a.** *Sketch of bottom-up and top-down approaches to cell motion. Bottom-up approaches study local mechanisms that drive cells, while top-down approaches study patterns of cell motion that are caused by local and global driving mechanisms. The dynamic morphoskeleton uncovers the centerpieces of cell trajectory patterns in space and time.* **1b.** *Snapshots of a simple analytic velocity field (blue) and its Lagrangian particle positions (green). The black dot marks the position of a particle that began at the black* $\boldsymbol{x}$ *marker at time 0. The frame-dependent velocity field suggests the presence of a vortical structure while the tissue undergoes exponential stretching. Figure courtesy of [6].*

A Lagrangian view that integrates over the history of cellular movements in time can clearly distinguish the way in which cells move apart or together, and ultimately provides a better perspective on cellular and tissue fate. This approach has successfully unraveled the complexity of passive fluid flows in problems that range from microfluidic mixing to atmospheric polar vortices using the notion of Lagrangian coherent structures (LCSs) [2]. Hyperbolic LCSs are time-evolving attracting and repelling manifolds that shape the overall spatiotemporal patterns in complex dynamical systems. In the context of morphogenesis, the *dynamic morphoskeleton* is the collection of attracting and repelling LCSs. From a practical perspective, we can compute these structures in terms of the largest finite-time Lyapunov exponent (FTLE), which is defined as

$$\Lambda_{t_0}^{t}(\boldsymbol{x}_0)=\frac{1}{|T|}\ln\left(\max_{\delta\boldsymbol{x}_0}\frac{\left|\overbrace{\nabla\boldsymbol{F}_{t_0}^{t}(\boldsymbol{x}_0)\delta\boldsymbol{x}_0}^{\delta\boldsymbol{x}_t}\right|}{|\delta\boldsymbol{x}_0|}\right).$$

Here, $\boldsymbol{F}_{t_0}^{t}(\boldsymbol{x}_0)=\boldsymbol{x}_0+\int_{t_0}^{t}\boldsymbol{v}(\boldsymbol{F}_{t_0}^{\tau}(\boldsymbol{x}_0),\tau)d\tau$ is the Lagrangian flow map that is induced

See **Morphodynamics** *on page 3*

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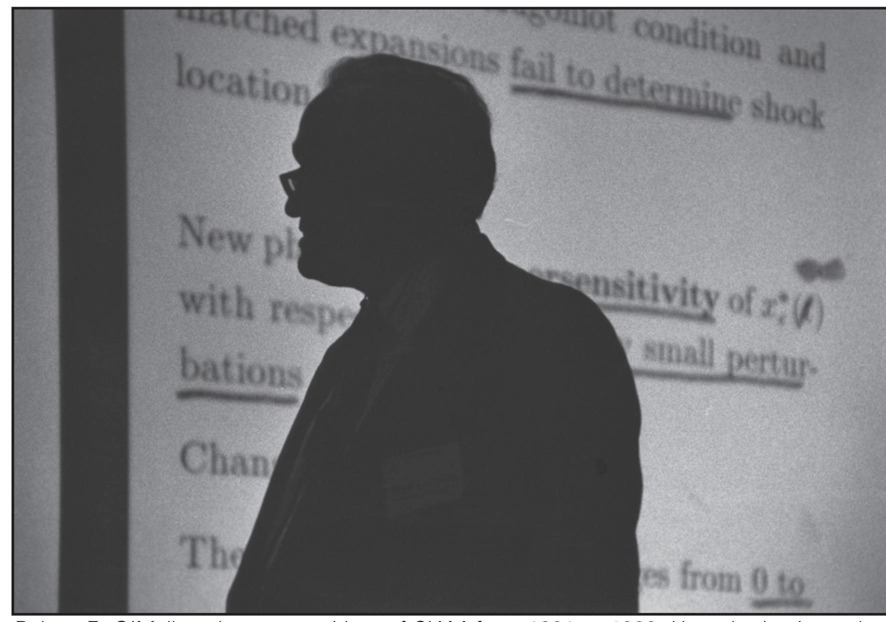
siam news
Volume 54/ Issue 3/ April 2021

Obituary: Robert E. O'Malley, Jr.

By Bernard Deconinck, Mark Kot, Randall J. LeVeque, and Ka-Kit Tung

It is with great sadness that we report that Robert E. O'Malley, Jr., who was president of SIAM from 1991 to 1992, passed away on December 31, 2020 at the age of 81. He was a world-renowned expert on singular perturbation problems, having written many impactful books and journal articles on the subject.

Bob was born on May 23, 1939 and grew up in Somersworth, NH. He began his higher education at the University of New Hampshire, where he earned a B.S. in electrical engineering in 1960 and an M.S. in mathematics in 1961. Bob received his Ph.D. in mathematics from Stanford University in 1966, where he wrote a dissertation on two-parameter singular perturbation problems under the supervision of Gordon E. Latta.



Robert E. O'Malley, Jr. was president of SIAM from 1991 to 1992. Here, he is pictured at a 1993 SIAM meeting. Bob was very fond of this photo and even had it hanging in his office.

Bob enjoyed a long and illustrious career in mathematics. After short appointments at the University of North Carolina, Bell Laboratories, New York University's Courant Institute of Mathematical Sciences, and the University of Wisconsin's Mathematics Research Center (where he met Candy, his wife), he returned to New York University in 1968. While there, Bob worked with Joseph Keller and other predominant researchers. This stint was followed by a one-year visit to the University of Edinburgh. Bob's lectures at these institutions formed the basis of his first book, *Introduction to Singular Perturbations*, which was published in 1974.

Robert E. O'Malley, Jr., 1939-2020. Photo courtesy of Katie Oliveras.

In 1973, Bob joined the Department of Mathematics at the University of Arizona; he founded Arizona's Program in Applied Mathematics in 1976. During this period, Bob worked extensively on singular perturbation problems in control theory. He was a forceful advocate for applied mathematics at Arizona and was especially supportive of young faculty.

Bob moved to Rensselaer Polytechnic Institute (RPI) in 1981, where he was Ford Foundation Professor, Chairman of the Faculty, and head of a Department of Mathematical Sciences that emphasized applied mathematics and computer science. Many years later, in 1999, Bob's colleagues at RPI hosted an O'Malley-fest:[1] a workshop on singular perturbations that brought roughly 60 mathematicians to Troy, NY, to celebrate Bob's 60th birthday.

At the end of 1990 and soon after a sabbatical to the Technical University of Vienna, Bob moved to the Department of Applied Mathematics at the University of Washington. He served as chair of the department and was the Graduate Program Coordinator for many years. He retired in 2009 but remained active as professor emeritus.

Bob was very productive during his time in academia and received many honors. He was a member of the inaugural class of Fellows for both SIAM and the American Mathematical Society. His SIAM Fellow citation[2] specifically mentions his contributions to asymptotics and singular perturbation theory. According to *Mathematical Reviews*, Bob authored 161 publications — including four books.

Bob was an extremely active member of SIAM and was president from 1991 to 1992. He also served as Vice President for Publications, sat on many editorial boards, and was involved in the organization of several meetings — most notably as Program Chair for the Second International Congress on Industrial and Applied Mathematics (ICIAM), which took place in Washington, D.C. in 1991. He was even SIAM's representative to the ICIAM Board in its formative years. In 2000, Bob became editor of the "Book Reviews" section of *SIAM Review*; he enjoyed this responsibility so much that he continued in the role through 2014, five years after his retirement. Thanks to his selections of books and reviewers, this section became one of the journal's most popular aspects.

Bob loved his work. He routinely came to campus even after he retired, and we had a better chance of running into him at the office than any other colleague. He loved differential equations, especially those that looked deceptively simple but presented examples or counterexamples of phenomena that he sought to understand. In fact, he was working on a new book about ordinary differential equations and collecting many such equations when he passed away. Throughout his career, Bob updated the techniques he used to approach these problems; he recently spent significant time on renormalization-based methods that originated in the physics literature.

Bob also loved books. He loved reading them, writing them, reviewing them, talking about them, and recommending them to others. His office was always filled with precariously-towering stacks of texts, particularly during his many years as editor of *SIAM Review*'s "Book Reviews" section. In addition, Bob loved exploring the history of mathematics. He enjoyed reading and writing about this history, and peppered his lectures with fascinating historical anecdotes. His 2014 book, *Historical Developments in Singular Perturbations*, combines his affinity for differential equations, history, and books; it likely contains the most complete bibliography on the subject.

Bob loved people. He knew everyone in the asymptotics and perturbation theory community, and many people far beyond it. He always kept an open-door policy and would gladly drop whatever he was doing to talk with whomever entered his office. Graduate students adored Bob, and many former students and colleagues would detour on their way through Seattle to visit him. He greeted everyone with his kind smile and made them immediately feel at ease during conversation. His broad range of interests allowed him to naturally connect with people. Bob often wrote personal notes of encouragement to students long after they had left the department. He was a wonderful colleague and will be sorely missed.

Bob's funeral mass took place at St. James Cathedral in Seattle, Wash., on January 9, 2021. His son Patrick gave an especially eloquent tribute about the ways in which Bob's mathematics opened up his family's world and brought the world to their door. Bob is survived by his wife Candy and his sons Daniel, Patrick, and Timothy.

Additional memories, recollections, tributes, and photos of Bob from students and friends are available online.[3]

Bernard Deconinck, Mark Kot, Randall J. LeVeque, and Ka-Kit Tung are faculty in the Department of Applied Mathematics at the University of Washington. All four interacted extensively with Bob O'Malley for many years.

[1] https://archive.siam.org/news/news.php?id=793

[2] https://www.siam.org/prizes-recognition/fellows-program/all-siam-fellows/o

[3] https://amath.washington.edu/news/2021/01/08/remembering-professor-bob-omalley

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SIAM Celebrates Mathematics and Statistics Awareness Month

By Jillian Kunze

April is Mathematics and Statistics Awareness Month![1] Each year, the Joint Policy Board for Mathematics—a collaboration between SIAM, the American Mathematical Society, the American Statistical Association, and the Mathematical Association of America—holds a month-long celebration to enhance public understanding and appreciation of mathematics and statistics. Both subjects have real-world societal impacts in nearly every imaginable field, such as medicine, biotechnology, energy, manufacturing, and business. Throughout the month of April, universities, high schools, student groups, research institutions, public information offices, and other related organizations host math-related activities. These activities often include workshops, competitions, departmental open houses, festivals, lectures, art exhibits, poetry readings, and other events centered around themes related to mathematics and statistics. Due to the ongoing pandemic, participants are encouraged to celebrate virtually this year and use the hashtag #MathStatMonth on social media to share their festivities.

[1] https://ww2.amstat.org/mathstatmonth/index.html

Mathematics and Statistics Awareness Month originated in 1986 as Mathematics Awareness Week under then-U.S. president Ronald Reagan, who noted that enrollment in U.S. mathematical programs was declining. Mathematics Awareness Week initially focused on national-level events, such as a mathematics exhibit at the Smithsonian Institution and a reception on Capitol Hill. It became Mathematics Awareness Month in 1999 and began to shift its emphasis towards local, state, and regional activities. In 2017, the name changed to Mathematics and Statistics Awareness Month to recognize important research in both fields. Though the number and breadth of celebrations have grown throughout the years, the event has remained dedicated to increasing the visibility of mathematical and statistical research across a wide audience.

University department chairs, high school teachers, public policy representatives, and other professional leaders can access and share resources that help educate the public about the importance of mathematics and statistics in ongoing scenarios like sustainability, internet security, disease, and climate change. Mathematical and statistical research drives technological innovation and leads to discoveries of broad societal importance across many scientific fields. Here, several members of the *SIAM News* Editorial Board detail the ways in which they utilize mathematics to solve engaging problems.

Hans Kaper (Georgetown University), editor-in-chief of *SIAM News*: "I am an applied mathematician by training and profession, and my interests lie in the mathematical modeling of natural phenomena. Mathematical modeling combines the laws of nature with observational data and expresses them into the language of mathematics. A mathematical model is an abstraction; it reflects reality but should not be thought of as "reality." It reveals the essential mechanism that drives the phenomenon of interest and allows us to explore various scenarios, whether theoretically or computationally.

Among the natural phenomena that have fascinated me in the past are reaction-diffusion phenomena in combustion systems, vortex formation and transport in high-temperature superconductors, and pattern formation in magnetic materials. I have recently become interested in climate issues and modeling the effects of human activities on Earth's climate system. Some of my current work is related to the dynamics of glacial cycles during the Pleistocene, problems that concern food systems and food security, and mathematical approaches to resilience."

Korana Burke (University of California, Davis): "Anyone who has tried to enter or exit a massive lecture room, music hall, or other large venue has likely been stuck in a crowd. This scenario is not a pleasant

See **Awareness Month** *on page 6*

Morphodynamics

Continued from page 1

by cell velocities. $\Lambda_{t_0}^{t}(\boldsymbol{x}_0)$ characterizes the maximum rate of separation of points in a neighborhood of $\boldsymbol{x}_0$—denoted by the infinitesimal vector $\delta\boldsymbol{x}_0$—during the time window $[t_0, t]$ (see Figure 2). The FTLE has a natural interpretation in continuum mechanics and is related to the largest eigenvalue of the Cauchy-Green strain tensor [7], therefore serving as a natural invariant measure of deformation in a continuous medium.

By choosing to integrate either forwards or backwards in time, one can delineate manifolds that are zones of attraction or repulsion over a given time interval; doing so establishes an organizing skeleton for the flow pattern (see Figure 2). The resulting dynamic morphoskeleton is frame invariant, easy to compute, and robust to noise and loss of local data because of its (integrated) averaging property [6].

When deploying this approach on light-sheet microscopy data from the embryo of a normally-developing chick that involves $\sim 10^5$ cells (see Figure 3a) [4], we see that the dynamic morphoskeleton consists of two repellers—critical boundary regions across which cells will likely assume different fates—and one attractor (see Figure 3b). Repeller 1 marks a dynamic boundary between the embryonic and extra-embryonic regions. By contrast, repeller 2 marks the anterior-posterior boundary of a characteristic feature that is known as the primitive streak (PS): a zone of strong cellular convergence (an attractor) during early embryogenesis. First, we note that repellers remain invisible to Eulerian and Lagrangian tools that researchers use in multicellular flows.[1] This fact may explain why repellers appear to be undocumented in the literature, despite their relevance for cell fate acquisition.

Second, we notice that just an hour after cells have barely started to move, the dynamic morphoskeleton captures the PS's early footprint — well before it is actually visible to conventional tools (see Figure 3c). This approach can also detect preliminary signatures of abnormal development. Use of a drug to inhibit the presence of a critical diffusible morphogen (FGF) that is required for early gastrulation causes the PS formation to fail (see Figure 3d) and results in a lack of anterior-posterior cell differentiation, which is quantified by the loss of repeller 2 (evident in the left panels of Figure 3b and 3d).[2] Similar analysis of a whole curved embryo of a developing fruit fly with roughly 6,000 cells allows us to visualize how the attractors and repellers characterize the motions that lead to gastrulation in both normal and pathological development [6].

[1] See the online version of this article for a corresponding animation.

[2] An animation in the online version of this article depicts the time evolution of the dynamic morphoskeleton for the FGF receptor inhibitor embryo.

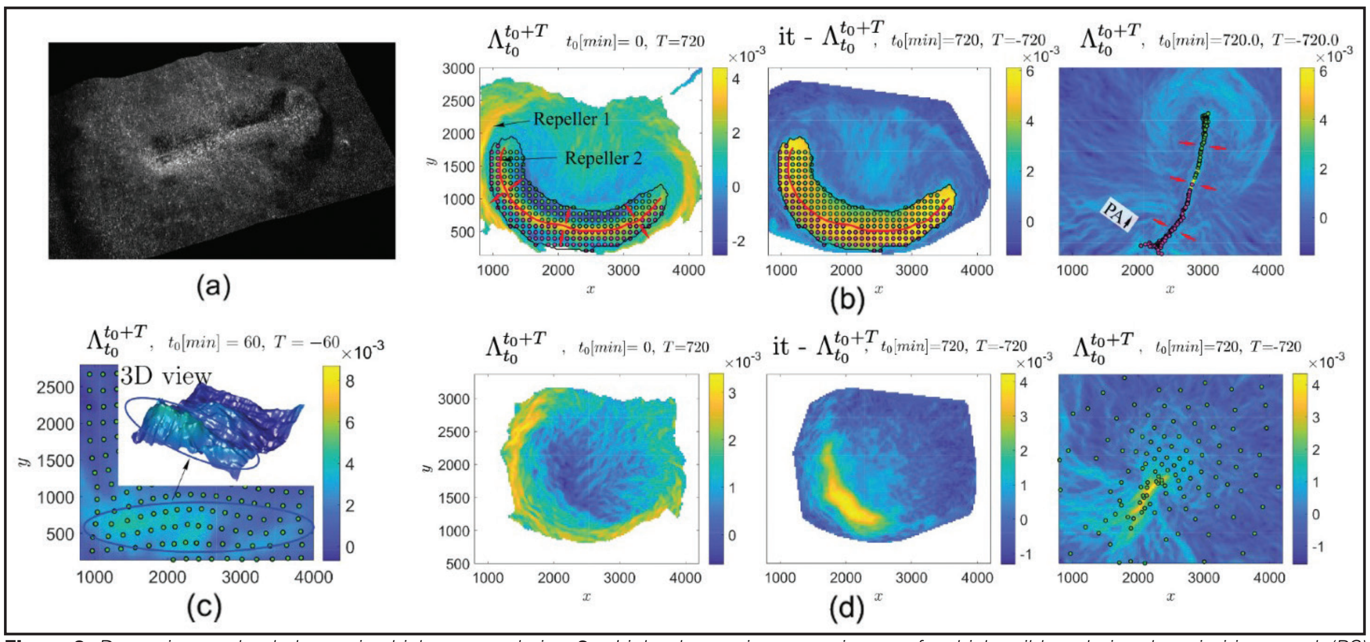

Figure 3. *Dynamic morphoskeletons in chicken gastrulation.* **3a.** *Light-sheet microscopy image of a chick epiblast during the primitive streak (PS) formation.* **3b.** *Left:* $FW\ FTLE_0^{12h}$ *highlights two repelling Lagrangian coherent structures (LCSs). Right:* $BW\ FTLE_{12h}^{0}$ *highlights the attracting LCS that corresponds to the formed PS. Center:* $BW\ FTLE_{12h}^{0}$ *field in the right panel is passively transported to the initial time; this marks the initial position of the mesendoderm precursor cells—bounded by the solid black line—that eventually form the PS. Cells that begin on different sides of repeller 2 form the anterior and posterior part of the PS. The finite-time Lyapunov exponent (FTLE) has units 1/min, and the axis units are in μm. The time evolution of the FTLE fields and cell positions for different* T *are available in an online animation. The lower panels of this animation depict the time-averaged velocity, cell positions, and a deforming Lagrangian grid.* **3c.** $BW\ FTLE_{1h}^{0}$ *ridge highlights the PS's early footprint (blue ellipse) using only the first hour of data when the cells (green dots)—initially released on a uniform rectangular grid—have barely moved.* **3d.** *The same situation as 3b for a chick embryo that is treated with a critical diffusible morphogen (FGF) receptor inhibitor. An online animation shows the time evolution of the FTLE fields and cell positions for different* T*. Figure 3a courtesy of Cornelius Weijer, 3b-3d adapted from [6].*

The enormous amounts of data that are rapidly becoming available from large-scale imaging of biological development require properly invariant methods of analysis. The aforementioned approach—which integrates local (cell-based) and nonlocal (neighborhood/tissue-based) cues—provides a step in the right direction. The dynamic morphoskeleton sets the geometric stage for uncovering the dynamic organizers of cellular movements and tissue form; it also provides a lens for uncovering the underlying relevant mechanisms at play. When we combine the morphoskeleton with the ability to track and manipulate gene expression levels, mechanical forces, etc., perhaps we will be able to determine the biophysical mechanisms that underlie normal and pathological morphogenesis and move a little closer to answering one of the grand questions of modern biology.

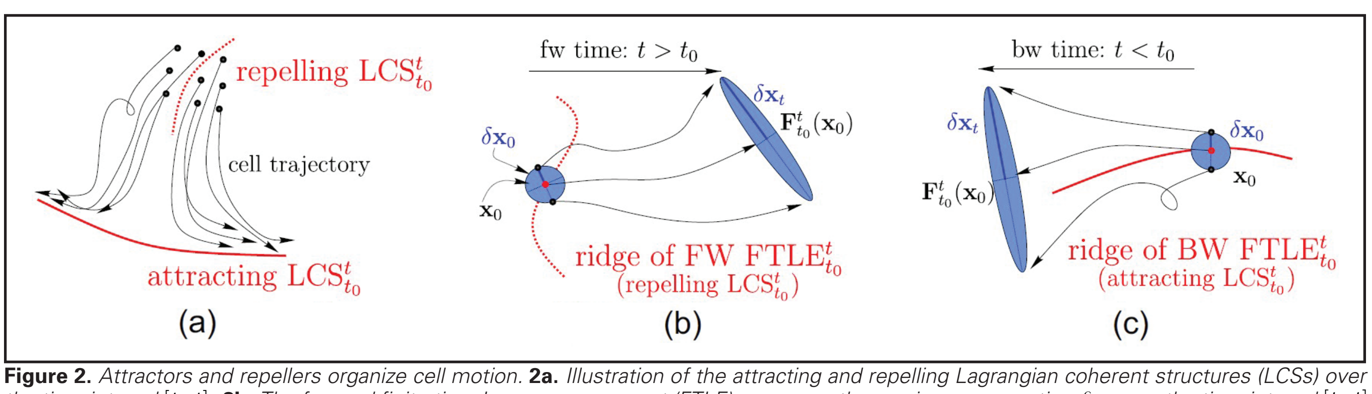

Figure 2. *Attractors and repellers organize cell motion.* **2a.** *Illustration of the attracting and repelling Lagrangian coherent structures (LCSs) over the time interval* $[t_0, t]$. **2b.** *The forward finite-time Lyapunov exponent (FTLE) measures the maximum separation* $\delta\boldsymbol{x}_t$ *over the time interval* $[t_0, t]$ *between two initially close points in the neighborhood* $\boldsymbol{x}_0$*. A forward-time FTLE ridge—a set of points with high FTLE values—marks a repelling LCS; nearby points from opposite sides of the ridge experience the maximum separation over* $[t_0, t]$, $t > t_0$. **2c.** *A backward-time FTLE ridge demarcates an attracting LCS — i.e., a distinguished curve at the final time that has attracted initially distant particles. Figure adapted from [6].*

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Stochastic Modeling for Weather and Climate Prediction

By Hannah Christensen

Regardless of the time or location, people seemingly always want to know the future weather. As late as the mid-20th century, the favored approach for weather forecasting involved *analogues* that were based on a large historical dataset of past weather reports. One would simply examine the record to find a day that was similar to the present day in question, then issue the historical evolution of the atmosphere as the forecast for the coming week. However, this method does not work in practice because the atmosphere is chaotic — its evolution is very sensitive to small details in the initial state. This was the central message of meteorologist Edward Lorenz's groundbreaking 1963 paper: analogue forecasting is doomed to fail since one simply cannot find a historical match to the current weather with sufficient accuracy [3].

Instead of using analogues, meteorologists now generate forecasts by combining the Navier-Stokes equations with equations that describe radiation, thermodynamics, water phase changes, and other phenomena in order to build a computer model of the atmosphere. Numerically solving these equations involves setting a discretization scale, which should be as fine as possible. However, we must also produce weather forecasts in a timely manner. Despite access to some of the world's largest supercomputers, this stipulation puts a hard limit on how fine the discretization scale can be. For weather forecasts that are out one or two weeks, this scale is around 10 kilometers. We must include the effects of all processes that occur below the discretization scale—including clouds, convection, and turbulence—in the model, but can only do so in an approximate manner via so-called "parametrization schemes." A key assumption is that one can successfully approximate the unresolved scales' impact on the resolved scale flow with a deterministic function of the resolved scales.

Two problems are immediately apparent. First, the Navier-Stokes equations show strong evidence of scaling symmetries [5]. In other words, if $\mathbf{u}(\mathbf{x},t)$, $p(\mathbf{x},t)$ is a solution to the Navier-Stokes equation, then $\mathbf{u}_r(\mathbf{x},t) = r\mathbf{u}(r\mathbf{x}, r^2t)$, $p_r(\mathbf{x},t) = r^2 p(r\mathbf{x}, r^2t)$ is also a solution for any scaling parameter $r > 0$. This scaling symmetry is consistent with the power-law behavior that is evident in atmospheric observations [6]. However, truncating the equations of motion at the discretization scale and replacing the unresolved scales in computer models with a deterministic parametrization scheme violate these scaling symmetries. Deterministic parametrizations essentially assume the presence of a spectral gap between resolved and unresolved scales, which does not exist in reality. The parametrization process is therefore a source of error in our forecasts. The second problem is that small-scale forecast errors will not remain confined to the smallest scales. Instead, they will exponentially grow in time and cascade upscale in space, thus causing our forecasts to diverge from the atmosphere's true evolution.

One solution to these two issues is to replace conventional, deterministic parametrizations with stochastic parametrization schemes [7]. We recognize that the grid-scale variables cannot fully constrain subgrid motions without a spectral gap. We therefore choose to describe the subgrid in terms of a probability density function (PDF) that is constrained by the resolved scale flow, then randomly draw from this evolving PDF to step our computer model forward. For example, instead of including the effects of the most likely arrangement of clouds, we include the effect of just one possible cloud field on the forecast's evolution.

To derive an appropriate form for the stochastic parametrization, we can characterize small-scale variability using high-resolution simulations that resolve the small-scale phenomena of interest. We do this by coarse graining these simulations before comparing them to a low-resolution forecasting model. Measurements of the "true" PDF of subgrid motions that are conditioned on the large-scale state not only provide further evidence that parametrization schemes should be stochastic, but also motivate the form of the stochastic parametrizations themselves [1] (see Figure 1).

To address the second aforementioned problem—the fact that small-scale forecast

See **Climate Prediction** *on page 5*

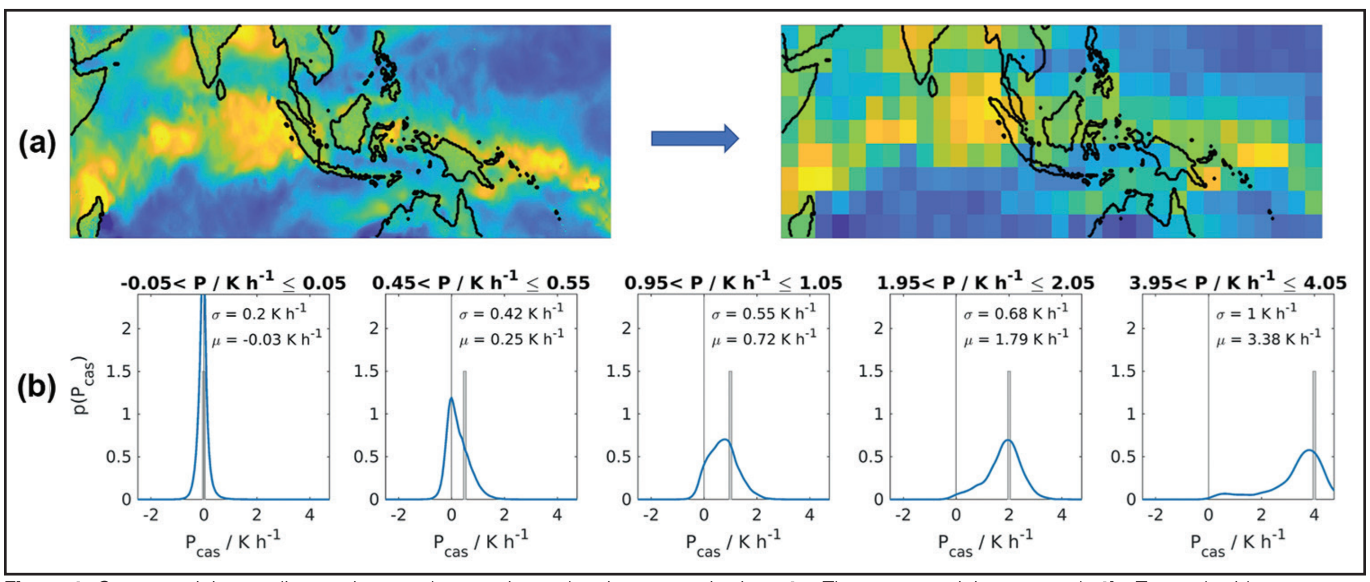

Figure 1. *Coarse-graining studies motivate and constrain stochastic parametrizations.* **1a.** *The coarse-graining approach.* **1b.** *True subgrid temperature tendency distribution (blue) compared to the estimate from a deterministic parametrization (grey rectangle and panel titles). Figure adapted from [1].*

Figure Skating

Continued from page 1

that describes the skater belongs to the field of *nonholonomic mechanics* [1]. This is in contrast to *holonomic mechanical systems*, which are defined by the property that allows all constraints to be written as a function of coordinates only. Studies of the fundamentals of mechanics, and thus our intuition about mechanics, mostly pertain to holonomic systems. Nonholonomic mechanics can be quite counterintuitive for an unfamiliar reader. For example, the famous Noether theorem of classical mechanics—which connects conservation laws with symmetries—is generally not valid in nonholonomic cases, except for energy conservation [5].

One of the earliest, most famous, and perhaps most pedagogical examples of a skater-like system is the Chaplygin sleigh [3, 4] — a flat, rigid object on ice with a fixed blade (a modern exposition is available in [1]). It is actually one of very few examples of integrable nonholonomic systems [7]. One can understand the Chaplygin sleigh as a two-dimensional model of a skater that lacks the ability to lean. When the position of the sleigh's center of mass and its moment of inertia are changing, the sleigh is akin to a figure skater who controls their motion by changing the position of their torso and limbs.

Previous researchers have explored the idea of controlling the Chaplygin sleigh motion by altering the position of its center of mass [8]. The question is: How can we control the sleigh to *produce a predetermined curve* on the ice, as if the sleigh were participating in a figures competition? For example, how can we achieve a typical shape from the figures competition in Figure 1a (on page 1)? The trajectory is not smooth at the "cusp" points. At these points, the skate's velocity with respect to the ice must be zero, and an experienced skater can perform a quick turn of the skate to continue the motion. To realize this motion in the Chaplygin sleigh, one can build up the trajectory from predetermined "patches"—e.g., pieces of a circle—with vanishing velocity at the ends and connect them with a quick turn at the cusp (see Figure 1, on page 1). Notice that no push is necessary at the cusp; the skater simply reverses the direction of motion from forward to backward or vice versa.

Things get more complex if one seeks to describe a skater's motion in three dimensions when there is a possibility for a side lean while skating. If the skater is *static* (i.e., not moving the parts of their body), this assumption allows a reduction of the mechanical system that describes the skater to a system of seven equations with seven unknowns: linear velocity of the skate; tilt angle; angular velocities of rotation about both the vertical and the blade; and three Lagrange multipliers that describe the reaction of constraints of ice contact, no forward tilt, and the nonholonomic constraint of the skating condition [6]. The system's mechanical energy is conserved, as expected with the absence of friction. If the projection of the skater's center of mass onto the blade coincides with the contact point (i.e., a balanced skater), two more constants of motion exist. We can understand one of them as the system's angular momentum around the vertical axis, and the second as having no physical relevance.

Given these three constants of motion, the system is integrable and constitutes an additional and highly nontrivial example of an integrable nonholonomic system [7]. If the skater is *unbalanced*—i.e., the projection of the center of mass on the blade's direction does not coincide with the point of contact—the motion is chaotic with a positive Lyapunov exponent of diverging trajectories. In terms of ice trajectories, the blade's motion on ice is either quasi-periodic in the integrable case (see Figure 2a) or chaotic in the non-integrable case (see Figure 2b). Just as with the Chaplygin sleigh, the linear velocity of the skate at the cusps goes smoothly through zero and the skater reverses their direction of motion.

It seems plausible to conjecture that experienced figure skaters adjust their positions to choose the integrable case for their performances, moving from one integrable case to another for better control. If correct, this conjecture can help us better understand the algorithms that our own minds use to control the motion — ultimately achieving a precise output in a motion as complex and non-intuitive as figure skating.

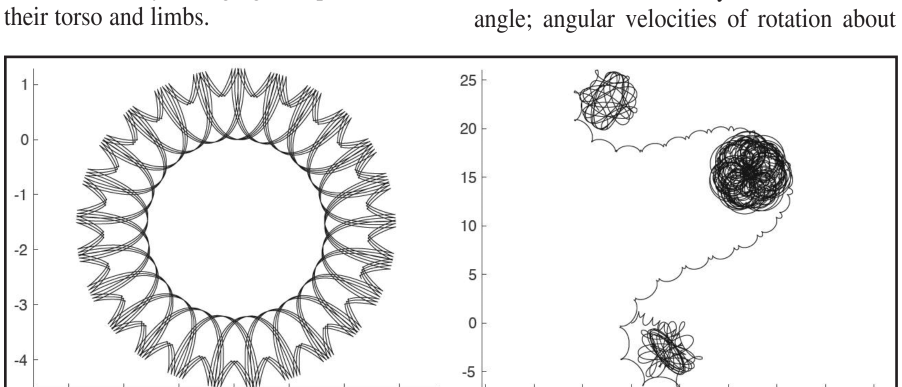

Figure 2. *Trajectories made by a static three-dimensional model skater on ice.* **2a.** *The integrable case, when the projection of the center of mass on the mass's direction of motion coincides with the contact point with the ice.* **2b.** *The non-integrable case, when the projection of the center of mass is away from the contact point. Figure courtesy of [6].*

This article is based on Vakhtang Putkaradze's invited presentation at the Second Joint SIAM/CAIMS Annual Meeting,[1] which took place virtually last year. Putkaradze received the 2020 CAIMS-Fields Industrial Mathematics Prize.

[1] https://www.siam.org/conferences/cm/conference/an20

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Meghan Rhodes is a Ph.D. candidate in applied mathematics at the University of Alberta who works on two fairly disparate topics: mathematically describing figure skating and modeling the mechanics of glioma spread. She used to figure skate seriously and still enjoys both real and theoretical skating. Vaughn Gzenda is a MSc student at Carleton University who is interested in geometric mechanics and control. When not studying control theory or playing a rock guitar, he is building a figure skating robot to extend the results of existing studies. Vakhtang Putkaradze is the Vice President of Transformation, Science, and Technology at ATCO. He was a Centennial Professor of Mathematics at the University of Alberta until 2019. Putkaradze works in applications of mathematics, particularly to problems of mechanics, nanotechnology, and energy. He is trying to supplement his subpar skating skills with geometric mechanics.

Climate Prediction

Continued from page 4

errors will not remain confined to the smallest scales—we transition from making a single forecast for an upcoming period to making a set of forecasts. The forecasts originate from different but equally likely starting conditions, which we estimate based on our measurements of the atmosphere. Each forecast also utilizes different random numbers in the stochastic parametrization scheme, thereby indicating various possible realizations of the small-scale processes. By skillfully accounting for all sources of error in our forecasts, we can ensure that they are reliable — i.e., statistically consistent with the observed evolution of the atmosphere. For instance, if we collect all of the days for which we predicted a 10 percent chance of rain, it should rain on 10 percent of those days. Stochastic parametrizations are clearly necessary; we cannot produce reliable forecasts without them.

Nowadays, however, we are not only interested in predicting the weather. Climate prediction is extremely important because it provides guidance for policymakers and enables a range of sectors to prepare for the future. But predicting the climate is a different problem than predicting the weather. In fact, Lorenz referred to weather prediction as a "prediction of the first kind" [4]. Such problems are initial value problems — the skill in the forecast comes primarily from accurate specification of the starting conditions and the system's resulting evolution away from these conditions. Climate prediction, on the other hand, is a "prediction of the second kind." In this context, we are interested in predicting a system's response to an external forcing. We cannot hope to predict the specifics of the weather on any given day, but rather seek to predict the weather's changing statistics.

Despite these differences, we produce climate predictions much like weather forecasts — though now we use a computer model of the *entire* Earth system, including the atmosphere, oceans, biosphere, and cryosphere, among other components. We also incorporate an estimate of how anthropogenic greenhouse gases and other emissions will evolve in the future—based on a range of policy-driven "emission pathways"—to assess possible forthcoming changes to the Earth's climate. The added complexity of a climate model, coupled with the need to produce predictions on century-long timescales, means that we must substantially coarsen the discretization scale to the order of 100 kilometers.

While the weather forecasting community has readily adopted stochastic parametrizations because of their measurable positive impact on forecast skill, the climate modeling community generally still uses deterministic models. However, our recent work demonstrates the potential of stochastic parametrizations to transform climate modeling much like they have transformed weather prediction. We show that the presence of stochasticity in climate models can alleviate long-standing systematic biases, such as mean state biases—like the distribution of precipitation [8]—and biases in modes of variability, like the El Niño–Southern Oscillation [2] (see Figure 2). Despite concerted efforts from the community, these stubborn biases in deterministic models have long resisted improvement.

Unpicking the way in which stochasticity leads to such dramatic improvements is nontrivial, and we generally must assess the mechanism for each phenomenon of interest. For example, while researchers can understand El Niño's basic existence as a deterministic coupling between atmosphere and ocean, its variability stems from high-frequency atmospheric wind stress forcing on the ocean surface. Simulations that include a stochastic parametrization reveal an improved distribution of atmospheric winds. In the Community Climate System Model (CCSM) and Met Office Unified Model (MetUM), the parametrization dampens an excessively active El Niño. But in the EC-Earth climate model, it enhances a too-weak El Niño (see Figure 2). If we assume that the underlying coupling strength between atmosphere and ocean differs among the various climate models, then a very simple delayed oscillator model of El Niño predicts this extraordinary result [9].

By improving the statistics of the fast "weather" in climate models, we enable the simulated Earth system to explore its attractor in a more realistic way and thus improve the model's fidelity. As I write this article, the International Panel on Climate Change is producing its Sixth Assessment Report.[1] This report will collate the state of the art in climate prediction and compare coordinated climate change experiments that were created with the world's leading climate models. For the first time, one of these models includes stochastic parametrization schemes. This is an exciting development, and I trust that many climate centers will soon adopt these techniques.

This article is based on Hannah Christensen's invited presentation at the 2020 SIAM Conference on Mathematics of Planet Earth,[2] which took place virtually last year. Christensen's talk is available on SIAM's YouTube Channel.[3]

1 https://www.ipcc.ch/assessment-report/ar6

2 https://www.siam.org/conferences/cm/conference/mpe20

3 https://www.youtube.com/watch?v=rtayL2JAjh8

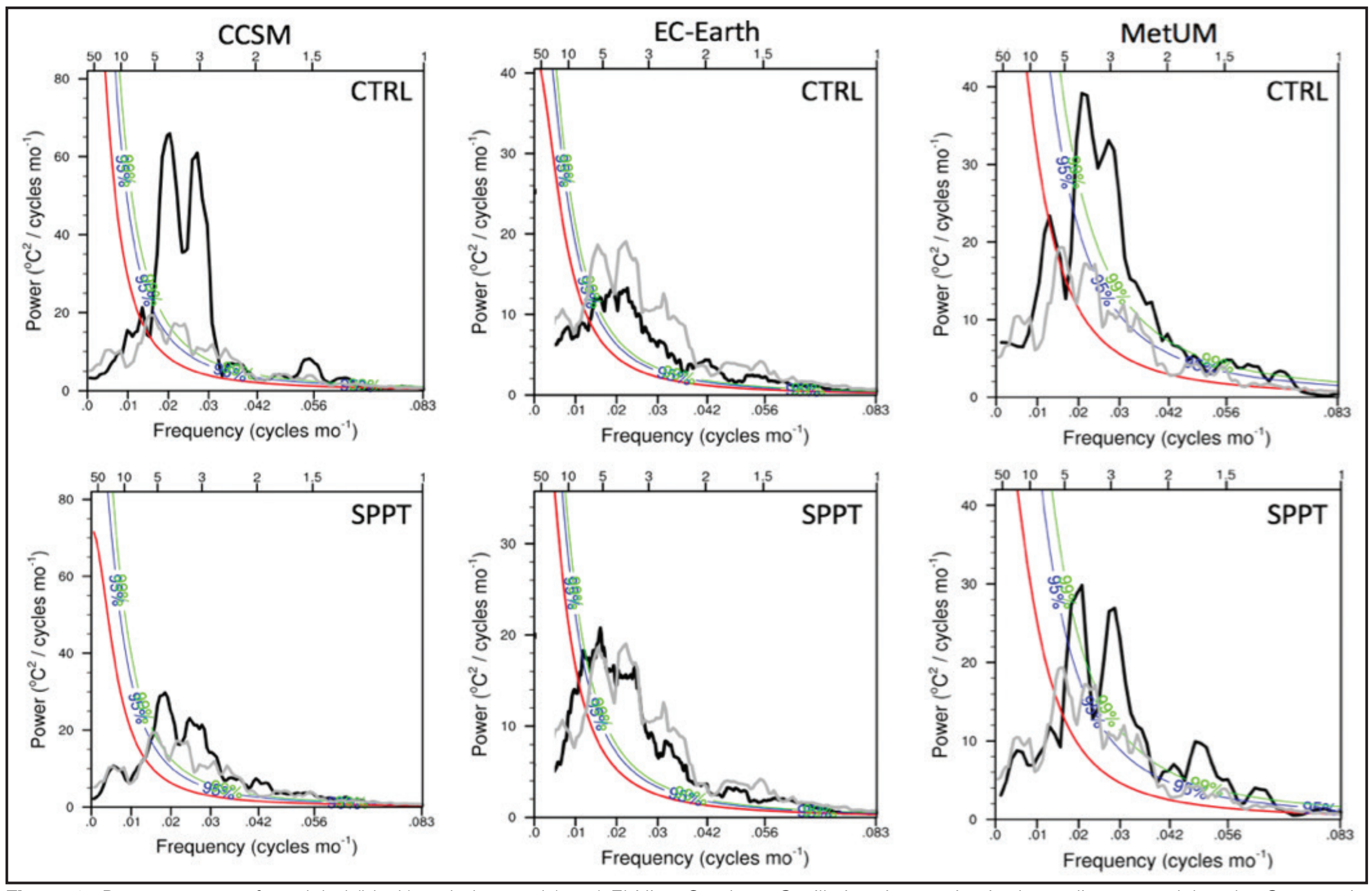

Figure 2. *Power spectra of modeled (black) and observed (grey) El Niño–Southern Oscillation time series in three climate models—the Community Climate System Model (CCSM), EC-Earth, and Met Office Unified Model (MetUM)—with and without the stochastically perturbed parametrization tendencies (SPPT). The power spectrum for each model with stochastic parametrization better matches the observed data. Figure adapted from [2] and [9].*

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Hannah Christensen is an associate professor in the Department of Physics at the University of Oxford and the David Richards Fellow at Oxford's Wadham College. Her research focuses on the role of fast atmospheric processes in the climate system, including convective storms, clouds, and turbulence.

A Mathematical Journey to Football

By Eric Eager

My love of numbers has always accompanied my love of sports. As a kid, I could recite the Pythagorean theorem as well as I could recount the number of catches that Jerry Rice made in each season of his hall-of-fame career. My childhood is annotated by my hometown Minnesota Vikings' heartbreaking losses. I knew all of the stats during any game and could recite those of others. This penchant served me well as I progressed in both my athletic and academic endeavors. I chose mathematics as my undergraduate major, a subject that came naturally to me and neatly coexisted with my love for playing and watching football as a college student at Minnesota State University Moorhead.

When it became noticeably clear that my career would likely not involve sports, I began taking the time to appreciate the beauty of mathematics and the learning thereof. I was hooked by real analysis, abstract algebra, differential equations, and the idea that math was a living, breathing thing that could model the world with the guidance of people like me. As a result, I opted to pursue my Ph.D. at the University of Nebraska–Lincoln (UNL).

At UNL, I grew to love all aspects of professional applied mathematics — from modeling to data analysis and simulation to theorem proofs. I developed models that led me into the discipline of mathematical biology, where I studied under world-class researchers and tackled important problems in population ecology, environmental biology, and gene regulatory networks. I also enjoyed conducting cutting-edge research about the scholarship of teaching and learning, and I mentored two Research Experiences for Undergraduates groups while in Lincoln. This work then took me to the University of Wisconsin–La Crosse, where I founded the Math Bio Working Group and received multiple grants from the National Science Foundation to mentor undergraduate research at the interface of mathematics and biology. I was actively accomplishing the goals I had set when I decided to become a mathematician many years earlier.

In 2015, my world changed. While in the middle of my academic career as a mathematician, I agreed to help a company called Pro Football Focus[1] (PFF) collect and analyze data for the National Football League (NFL) and college football. PFF found its way into my world through my weak interests in fantasy football and the plight of my favorite team, the Kansas City Chiefs. The organization was collecting and analyzing data in a way that I had never seen. "Moneyball for football," I thought. I was already watching these games religiously, so I figured that getting paid to do so would keep everyone in my family happy. My mother thought I was wasting my time.

CAREERS IN MATHEMATICAL SCIENCES

At the time, researchers were doing little in the way of mathematical or statistical analysis with this type of information. This fact surprised me but ended up being extremely advantageous. I knew that nearly all of the NFL teams were paying PFF for its services; here was an opportunity for me to use my skills as an applied mathematician to finally make a difference in the game I loved so much. I was learning more about football and data science than I ever thought possible and pushing my professional capabilities forward with each day.

By 2018, my colleague George Chahrouri and I had developed enough quality football-based machine learning models for PFF CEO Neil Hornsby and former Cincinnati Bengal and PFF majority owner Cris Collinsworth to offer us the job of a lifetime: the opportunity to work full time in football as data scientists for the world's most prominent football data company. While it was not easy to leave my position at UW–La Crosse, especially because I had just earned tenure the previous year, it was a risk I decided to take.

Now almost three years later, I have held several different roles at PFF as the company continues to grow. From data scientist to Vice President of Research and Development,[2] I have consulted with teams on the use of our data to evaluate players, coaches, and front office members. I have also mentored other employees who have grown to do the same thing. When PFF moved from a data provider to an analysis company, our group began to build machine learning models for fantasy football and simulators for gambling. Our dashboards and contract projections help pair agents with rising professional prospects, and our text and video content provide alternatives to traditional sports media. I have been lucky enough to appear on NFL Network and frequent talk radio shows in almost every major media market in the country to discuss fantasy football, gambling, and the NFL draft. PFF's work has been featured on NBC Sunday Night Football as well as the TODAY show. In fact, MSNBC's national political correspondent Steve Kornacki used our simulation to analyze the playoff picture in the same engaging way he analyzed the electoral map during the 2020 U.S. election season.

My typical workday is never typical, and my mathematics training routinely comes into play through the habits of mind that are necessary to navigate the competitive and ever-changing world of sports analytics. While PFF's data set was already immense when I began working with the company in 2015—and even more so when I joined full time in 2018—the data we use to better understand the game of football continues to grow in both rows and columns each day. Staying up to date with the newest methods of generating insights from this data is no different than being privy to the latest theorems or models in mathematical biology. I therefore spend much of my time reading the work of other analysts, including those within and outside my group, to determine whether I can integrate any of their ideas into my models. Does incorporating the continuity of a team's offensive line reduce the errors in our fantasy football projections? Can adding the speed of a pass rusher's first move off the line of scrimmage help sharpen our ability to employ machine learning to evaluate his talent level? What is the best way to utilize subject matter expertise to communicate our findings to stakeholders and ensure that the information will be used?

Throughout the course of both my own education and that which I gave my students, the narrative always remained that one can do anything with a mathematics degree. There were times when I questioned such a notion, but ultimately my career path has convinced me that it really is completely true. The marketplace for sports analysts is quickly evolving but far less rigid than most. With so much public data readily available, your resume is the insight that you present to the world; do not be afraid to get started and share your work!

Eric Eager is the Vice President of Research and Development at Pro Football Focus (PFF), a worldwide leader in data and analytics. Prior to joining PFF, he was an applied mathematician who studied mathematical biology, ecology, and the scholarship of teaching and learning.

Eric Eager (left) of Pro Football Focus (PFF) previews Super Bowl LIV between the Kansas City Chiefs and the San Francisco 49ers in Miami, Fla., with Soren Petro of Sports Radio 810. Photo courtesy of Soren Petro.

[1] https://www.pff.com

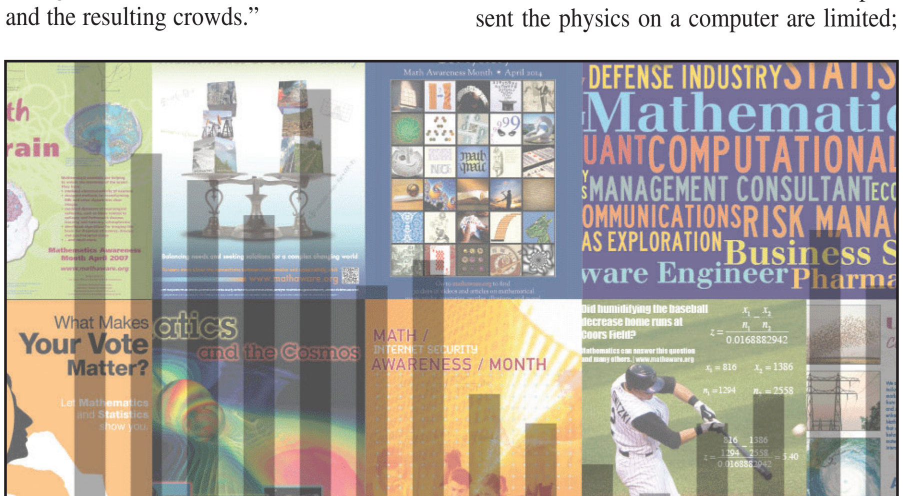
[2] https://www.pff.com/analyst/eric-eager

Awareness Month

Continued from page 3

feeling in the best of times, and the ongoing pandemic has taught us that it can even be a health hazard. We can utilize differential equations to model the way in which people adjust their movements when they are in a crowd. These models allow us to study the appearance of bottlenecks and also their resolution. The results can generate more efficient entering and exiting protocols for existing venues and lead to the design of new venues whose geometry and seating arrangements would minimize bottlenecks and the resulting crowds."

Oana Marin (Argonne National Laboratory): "Differential calculus revolutionized mathematical physics at the end of the 17th century. Almost two centuries later, computers have provided a platform for the simulation of any differential equation. Although the progress of science may now seem boundless, the expectations have shifted. We wish to explore all of the physical scales, we hope to simulate physical reality in real time on our mobile phones, and we want computers to teach us everything that we cannot analytically derive ourselves. Nowadays, we are disappointed to discover that the numerical methods we use to represent the physics on a computer are limited; computers themselves are not as reliable as forethought because of computational round-offs, sophisticated heterogeneous supercomputers, and so forth. The work of numerical analysts is almost akin to revisiting the days of Gottfried Wilhelm Leibniz and Isaac Newton, but this time equipped with a great weapon: the supercomputer."



Mathematics and Statistics Awareness Month recognizes the value of mathematical and statistical research in the solution of problems with broad societal reach.

Lois Curfman McInnes (Argonne National Laboratory): "Computational science and engineering (CSE)—which unites mathematics and statistics, computer science, and core disciplines from the sciences and engineering—is actively transforming discovery and innovation in essentially all areas of science, engineering, technology, and society. CSE has become the essential driving force for scientific progress when classical experiments or conventional theory reach their limits, and in applications where experimental approaches are too costly, slow, dangerous, or impossible. Advanced mathematics-based computing is actively inspiring a wide range of scientific discoveries, better designs for new products, and support for decision-makers. Next-generation opportunities are moving beyond interpretive simulations and toward predictive science."

Ali Pinar (Sandia National Laboratories): "I have been part of many interdisciplinary efforts throughout my career. In all cases, mathematics has been the common language that enabled communication between different disciplines. And the success of introducing rigor to a new area has been directly proportional to the ability to adopt mathematics."

Rosemary Renaut (Arizona State University): "These days, my research involves the solution of inverse problems. I like to use the example of a speed camera that reads your license plate. The camera obtains a blurred version of the plate, but mathematics is at play in generating the letter that comes with the ticket by restoring the image from its blurred form. More complicated inverse problems surround us in many fields. For instance, consider medical images from PET and MRI scanners that assist with the diagnosis of various complications. Mathematicians have again been pivotal in the design of robust and accurate approaches that reconstruct the images from measurements that engineers and physicists cleverly obtain. We are not just "number people" — far from it. We are engaged in exciting problems with huge societal relevance that motivate our current research directions."

SIAM encourages its members to partake in Mathematics and Statistics Awareness Month and proudly share the importance, beauty, and applicability of their research with the general public.

Jillian Kunze is the associate editor of SIAM News.

Mathematics in Industry: What, When, and How?

By Mitchel Colebank

Graduate students across the fields of science, technology, engineering, and mathematics (STEM) routinely face an important post-degree career decision: academia or industry? Unfortunately, companies rarely provide the title of "mathematician" in job postings, which can make it difficult for new Ph.D. candidates in applied mathematics to fully understand the day-to-day responsibilities of industry-based positions. But recent years have seen increased conversation and guidance about the importance of applied mathematics in business, industry, and government (BIG) settings. Furthermore, the boom of "big data" has created numerous quantitative science jobs in organizations that specialize in healthcare, medicine, government security, and defense research. Given the wide range of current opportunities, applied mathematics graduates must think about which career paths are best suited for them.

In 2018, SIAM published *BIG Jobs Guide: Business, Industry, and Government Careers for Mathematical Scientists, Statisticians, and Operations Researchers*[1] to help students and early-career mathematicians understand the growing industry job market. Authors Rachel Levy, Richard Laugesen, and Fadil Santosa agree that the most common job titles for mathematicians include "data scientist," "analyst," and "software engineer," though other "quantitative"-based titles are also abundant. It is thus wise for students to market themselves as more than simply mathematicians. "One misconception is that you can present yourself as a mathematician and people will automatically know what you bring to the table," Levy said. "Job seekers should practice describing the types of problems that interest them and the ways in which they have tackled a problem with approaches that might be relevant to a prospective employer." Another common misconception is that industry jobs lack intellectual challenge or are focused on simple "number crunching" tasks. Laugesen offered an alternative viewpoint. "The problems in industry and government tend to require a broad range of expertise," he said. "The mathematical scientist must therefore interface with team members who possess quite different conceptual toolboxes."

One of the starkest differences between academic and BIG settings is the quantity of open positions. While most major metropolitan areas—and many smaller regions as well—typically offer ample satisfying job prospects in industry, academia is less straightforward. Early-career academic professionals often must accept what is available, rather than target specific areas. Moreover, BIG jobs generally have clear-cut working hours, whereas the burden of teaching, advising students, submitting grant proposals, and conducting personal research can greatly exceed the 40-hour workweek in academia. However, academic jobs frequently provide more intellectual freedom for mathematicians to pursue their own research endeavors, while BIG positions are commonly created and funded to address specific problems.

CAREERS IN MATHEMATICAL SCIENCES

Switching from Industry to Academia

Though the switch from industry to academia might be uncommon, it is certainly possible. Laura Ellwein Fix, an associate professor of mathematics at Virginia Commonwealth University, spent several years in industry before returning to school and pursuing her Ph.D. in applied mathematics. "My job as a quality engineer was not what I expected," Ellwein Fix said. "I thought it would entail more long-term scientific problem solving, but I found that needs in manufacturing were primarily customer-driven firefighting."

Based on both her prior interest in industry and Ph.D. research in mathematical biology, Ellwein Fix completed an internship at a large pharmaceutical company while doing graduate work. Although the experience was both engaging and enjoyable, she ultimately accepted a postdoctoral position in bioengineering at Marquette University before securing an academic appointment in mathematics. "The reward of deriving mathematical bases for scientific phenomena while being able to conduct outreach and teach young professionals solidified my desire for an academic position," she said. Ellwein Fix appreciates the freedom to chase her own research goals. She also commented on the varying time flexibilities of industry versus academia. In her industry job, Ellwein Fix was given brief timelines to complete projects and allotted only a short period to brainstorm and problem solve. Her academic position, however, permits a semi-flexible schedule that allows her to spend the appropriate time addressing new ideas in mathematical physiology while still meeting reserach, teaching, and administrative deadlines and maintaining a healthy work-life balance. The switch from industry to academia was therefore beneficial for Ellwein Fix, as it provided insight into both job markets that helps her advise undergraduate and graduate students on their career goals.

Figure 1. *HeartFlow cofounder and chief technology officer Charles Taylor reviews a HeartFlow Analysis three-dimensional model. Figure courtesy of HeartFlow.*

Switching from Academia to Industry

Conversely, switching from academia to BIG is more common. Charles Taylor left his position as an associate professor at Stanford University to cofound HeartFlow,[2] a company at the intersection of computational modeling, artificial intelligence, and

See **Mathematics in Industry** *on page 12*

[1] https://my.siam.org/Store/Product/viewproduct/?ProductId=29783110

[2] https://www.heartflow.com

BLIS: BLAS and So Much More

By Field Van Zee, Robert van de Geijn, Maggie Myers, Devangi Parikh, and Devin Matthews

The Basic Linear Algebra Subprograms (BLAS) have profoundly impacted scientific software development. This application programming interface (API) supports basic linear algebra functionality, including vector-vector operations like dot product and axpy, matrix-vector operations like matrix-vector multiplication and rank-1 updates, and matrix-matrix operations like matrix-matrix multiplication and solutions for triangular systems with multiple right-hand sides. It enables portable high performance and a level of abstraction that upholds the development, maintenance, and readability of high-quality applications. Here we discuss the BLAS-like Library Instantiation Software (BLIS),[1] which facilitates rapid instantiation of BLAS and BLAS-like operations and provides both the original interface as well as alternatives [9].

Scientists have long expected hardware and middleware vendors to deliver high-performance implementations of BLAS; until the mid-1990s, only these proprietary solutions were available. The open-source Portable, High-Performance ANSI C (PHiPAC) project [2]—and later the first widely-used Automatically Tuned Linear Algebra Software (ATLAS) implementation [10]—included important developments, such as the achievement of reasonably high performance via coding in C, autogenerating code, and autotuning parameters. In the late 1990s, Kazushige Goto introduced a new way of blocking matrix-matrix multiplication that took advantage of two levels of cache memory in central processing units [3]. At the time, this method improved performance by up to 10 percent over ATLAS and existing vendor libraries. Goto's approach—now called the Goto algorithm—became the foundation for GotoBLAS and still forms the basis for most of the highest-performing BLAS implementations, both proprietary and open source. His implementation lives on as a fork of GotoBLAS called OpenBLAS.

One drawback of GotoBLAS was that much of the source code targeted specific architectures in addition to specific BLAS operations. BLIS restructures Goto's algorithm so that this low-level, machine-specific code is limited to the implementation of a few "microkernels" within five levels of generic cache- and register-blocking loops (see Figure 1). Once the microkernels are implemented, all BLAS and BLAS-like functionality becomes instantiated and is ready for use. As with Goto's algorithm, the rearrangement or packing of data at strategic points within the algorithm improves data locality and reduces the number of translation lookaside buffer (TLB) misses. It is important to note that analytical models, rather than empirical tuning, determine the various parameters in BLIS [5]. This approach yields a more flexible, portable, and easily maintained framework that achieves nearly optimal performance, as reported on the BLIS GitHub repository performance pages.[2]

The native interface that BLIS supports is object based, meaning that attributes like matrix domain (real versus complex), precision (half, single, double, or extended), dimensions, and storage format are hidden. BLIS also provides a native "typed" interface that resembles BLAS while offering extensions for various features, including the specification of separate row and column strides for each matrix operand. Finally, BLIS contains a BLAS compatibility interface that implements the conventional BLAS API. Users can easily create their own interfaces that map to either of the underlying BLIS interfaces (object or typed).

SOFTWARE AND PROGRAMMING

BLIS does not only export top-level interfaces; it also exposes lower-level components that researchers can compose in innovative ways. For example, BLIS can mix the domains and precisions of operands [8]. To support this feature, it incorporates the necessary conversions into the packing component that already forms a part of the basic algorithm. This technique has also been employed for a separate library for tensor contraction called TBLIS [6], which avoids the necessary rearrangement of data that occurs when such operations are cast in terms of traditional BLAS. In addition, BLIS has enabled the practical implementation of Strassen's algorithm [4]. These examples demonstrate that BLIS is as much a conceptual toolbox for innovation as a tangible framework for code instantiation.

BLAS-like Library Instantiation Software (BLIS): Working at the intersection of algorithms and architecture. Photo by Robert van de Geijn and Maggie Myers.

Since its debut on GitHub in 2012, BLIS has amassed a sizeable online community that includes volunteers who maintain packages for various distributions of Linux and other operating systems. Supported operating systems include Debian Linux, Ubuntu Linux, Gentoo Linux, Extra Packages for Enterprise Linux/Fedora Linux, openSUSE Linux, and GNU Guix. BLIS also offers macOS and Conda support, along with limited support for Windows dynamic-link libraries (DLLs). Users may download and compile the source code themselves via either `git`-cloned repositories or source snapshots in `.zip` or tarball formats. In addition to the "vanilla" distribution of BLIS that The University of Texas at Austin manages (in collaboration with academic, industry, and community partners around the globe), Advanced Micro Devices, Inc. (AMD) maintains[3] a separate fork of BLIS that contains optimizations that are specific to Zen-based microarchitectures, as well as enhancements that their corporate customers request. An annual workshop called the BLIS Retreat[4] provides a forum for many stakeholders from academia, industry, and government laboratories to exchange ideas and discuss the latest research.

While creating an entire framework—with support from the National Science Foundation and industry gifts—has taken the better part of a decade, the techniques that underlie BLIS are remarkably simple. During a four-week massive open online course (MOOC) titled "LAFF-On Programming for High Performance,"[5] we examine BLIS's matrix-matrix multiplication to illustrate important issues in high-performance computing. This course, which is part of a series of MOOCs,[6] attracts both novices and experts and lowers barriers for entry into the field — much like BLIS lowers barriers to porting BLAS-like operations to new environments and architectures.

BLIS continues to grow by rapidly incorporating support for emerging architectures and instruction sets—such as Intel® Advanced Vector Extensions 512, AMD EPYC, ARM Scalable Vector Extension, and IBM POWER10—and addressing dense linear algebra operations beyond the traditional BLAS interface. The simplicity of BLIS's underlying techniques and concepts has enabled a myriad of improvements. However, the BLIS framework—as a concrete instantiation of these ideas—has begun to reach certain limits. Instead, alternative one-off instantiations of the BLIS concept have enabled efficient tensor contraction [6], machine learning primitives [11], and even operations that are relevant to biostatistics [1]. But is it necessary to look beyond BLIS to address these exciting problems? We believe that the answer is "no" — or rather, the next great adventure for BLIS is to make it so.

This article is based on Robert van de Geijn's invited talk at the 2020 SIAM Conference on Parallel Processing for Scientific Computing,[7] which took place in Seattle, Wash., last year. Van de Geijn, Van Zee, and others received the 2020 SIAM Activity Group on Supercomputing Best Paper Prize for their paper on the BLIS framework [9]. Van de Geijn's presentation is available on SIAM's YouTube channel.[8]

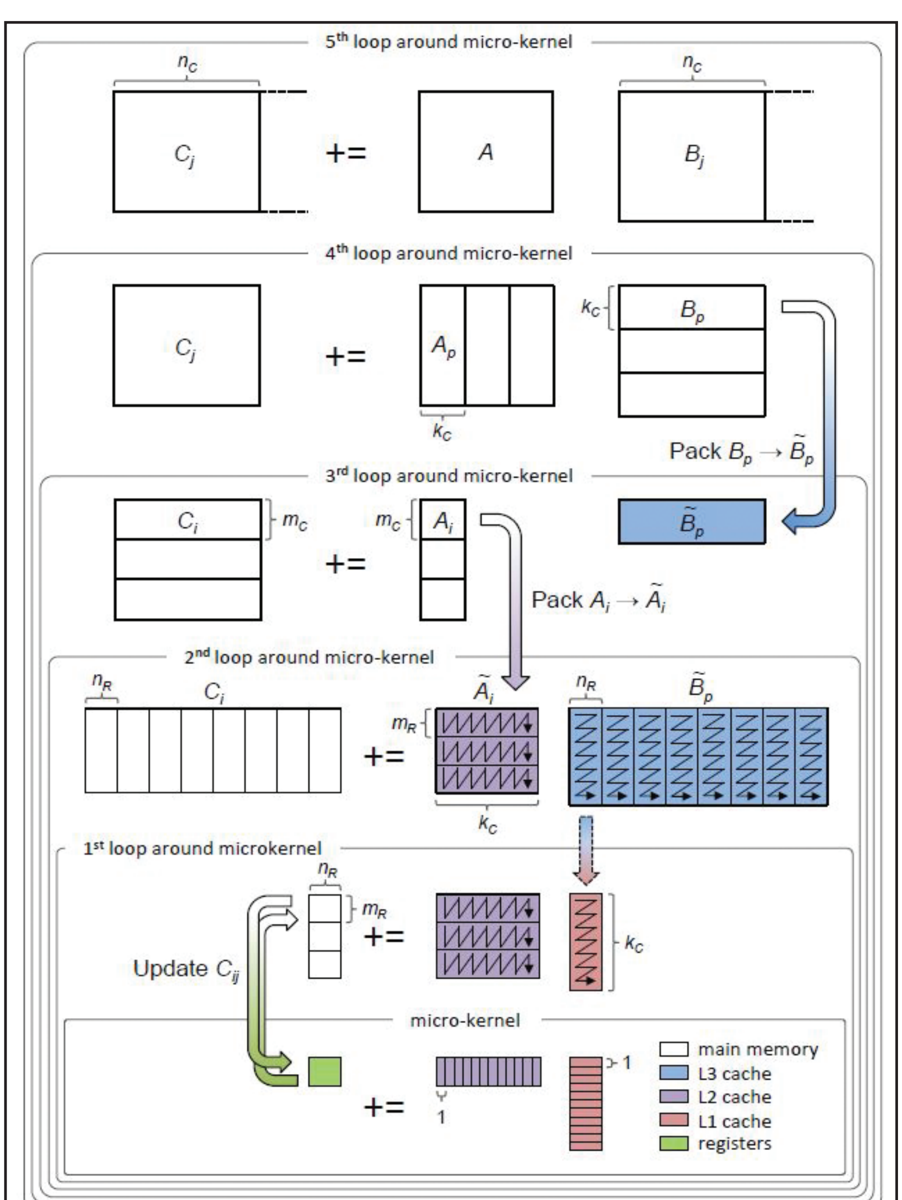


Figure 1. *The BLAS-like Library Instantiation Software (BLIS) refactoring of the GotoBLAS algorithm as five loops around the microkernel. Image courtesy of [7].*

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See **BLIS** *on page 10*

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[4] http://cs.utexas.edu/users/flame/BLISRetreat2020

[5] https://www.edx.org/course/laff-on-programming-for-high-performance

[6] http://ulaff.net

[7] https://www.siam.org/conferences/cm/conference/pp20

[8] https://www.youtube.com/watch?v=1biep1Rh_08

When Contagion Rules

BOOK REVIEW
By Paul Davis

The Rules of Contagion: Why Things Spread – and Why They Stop. *By Adam Kucharski. Basic Books, New York, NY, July 2020. 352 pages, $30.00.*

A book about pandemics that came off the presses just as COVID-19 upended life as we knew it? Precious little about COVID itself? Terrible timing? Completely irrelevant?

Wrong on all counts. Adam Kucharski's *The Rules of Contagion* is a deeply informed and widely accessible account of the evolution and breadth of model-based epidemiology. The subtitle proclaims the book's scope: *Why Things Spread – and Why They Stop*. All of these "things" are timely; some are humorous, most are frightening. They include misinformation on the internet, gun violence in cities, viral tweets, financial crises, and the spread of genetic variants of diseases, including highly contagious COVID variants.

Kucharski, an associate professor and a Sir Henry Dale Fellow in the Department of Infectious Disease Epidemiology at the London School of Hygiene & Tropical Medicine, writes about—and works in—a data-driven side of applied mathematics that has taken center stage in the last year. The field is a perfect microcosm of the larger world that many SIAM members inhabit: mathematics, modeling, data, and computation, blended in whatever proportions best suit the moment's challenge.

Kucharski writes with a sense of presence—an awareness of unknown outcomes and possible alternatives—not with biblical certainty. He tells stories with the authority and personal insight of a feet-on-the-ground participant who is balanced by the judgment and perspective of experience.

In the first chapter, Kucharski offers "A Theory of Happenings" — a sketch of the origins of epidemiological models that is free of equations and jargon. He describes mechanistic models that arose naturally to answer questions beyond the reach of experiments, such as "Can malaria be stopped without killing every mosquito?" As they evolved, the models revealed even more insight. The best, perhaps, was the possibility of herd immunity — the first of many points where Kucharski's story touches today's ongoing COVID-19 crisis.

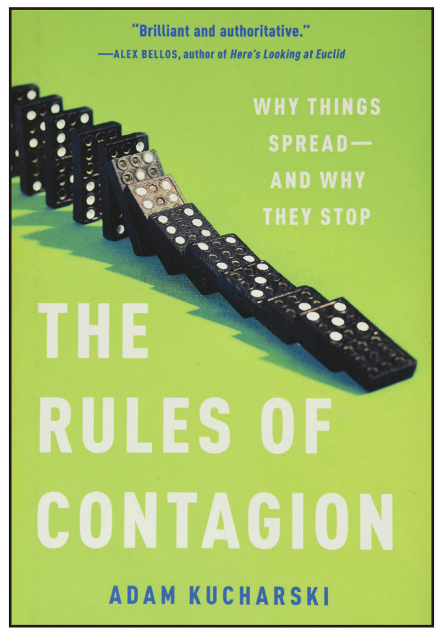
These models were ultimately able to answer other questions as well. In 2015, the island of Martinique faced an outbreak of the mosquito-borne Zika virus. With Zika came Guillain-Barré syndrome, a muscle-weakening immune disorder that can gradually paralyze a victim (and had coincidentally threatened Kucharski throughout his childhood). What was the likely shape of the outbreak curve? Would the small island's supply of eight ventilators suffice for a slow, flat outbreak or be overwhelmed by a rapid, sharp spike? Data-driven modeling predicted a slow, flat outbreak. In the end, no more than five patients at a time ever required the use of ventilators.

Kucharski's allegiance to data is apparent throughout the book. Some SIAM members might be startled to hear his suggestion that the papers of Alfred Lotka and others led outbreak analysis "away from real-life epidemics."[1] Kucharski goes on to explain that two decades after Lotka, mathematical epidemiologist Klaus Dietz brought "the theory of epidemics out of its mathematical niche and into the wider world of public health" when he introduced the reproduction number R: the average number of new infections that one infected person is expected to generate. Dietz recovered this powerful idea from a paper by malaria researcher George MacDonald.

Of course, R and its subscripted relatives have been prominently featured in the news since the arrival of COVID-19. *SIAM News* readers will likely appreciate Kucharski's subsequent description of the four factors that influence R. He calls these factors "DOTS" — Duration, Opportunity, Transmission probability, and Susceptibility. Here the rubber of mathematical modeling meets the twisting road of real-life data.

[1] And perhaps those readers will recall that "One's meat is another's poison."

The subsequent chapters are filled with Kucharski's accounts of epidemiological models' insights into a wide swath of human endeavors and social challenges, particularly those enabled by networks. Social influence, computer viruses, and assaults on privacy are but a few examples. R and DOTS are only two of many perspectives from which to penetrate the mysteries of these branching, diverging, and intertwining plot lines. The pleasures of reading *The Rules of Contagion* are the insights that result from artful and informed mathematical modeling; the worries are the threats of the malicious outbreaks under study.



The Rules of Contagion: Why Things Spread – and Why They Stop. By Adam Kucharski. Courtesy of Basic Books.

Kucharski's ably-told stories ignite such worries around the first computer worms, the Stuxnet worm that took control of Iranian uranium centrifuges, and the household Bluetooth devices that were surreptitiously commandeered to power denial-of-service attacks. Simple bots and deceptive websites notwithstanding, Kucharski warns that "when it comes to online manipulation, it turns out that something much subtler—and far more troubling—has been happening." False information from fringe websites can be laundered through legitimate news outlets "just as drug cartels might funnel their money through legitimate businesses to hide its origins."

Kucharski analyzes these network-enabled threats with the same informed perception he brings to contagious diseases and social ills. He demonstrates that understanding the relevant rules of contagion leads to strategies that defend against such threats. For example, some countermeasures to outbreaks of misinformation might "work by targeting different aspects of the reproduction number," while studies of online contagion have indicated the significance of broadcast events that amplify content. Events that disseminate misinformation are therefore points of attack, much like adult mosquitos are for malaria. After an outbreak of misinformation, mathematical models can estimate the preventive effects of various mitigation efforts. Modeling this kind of contagion builds a framework for both development and assessment of policy options.

Kucharski argues persuasively for making the components of the online information ecosystem—including citizens, media outlets, political organizations, and social media platforms—"more resistant to manipulation." This complex process begins with a thorough understanding of contagion to avoid the risks that are associated with potentially blaming the wrong source or proposing overly simplistic moderation strategies; e.g., "bad air" was once thought to cause malaria. And some people have blamed masks for causing COVID.

Kucharski's discussion of "Tracking Outbreaks" in the book's penultimate chapter directly connects to a present concern: the fear that the COVID-19 virus will evolve to outwit vaccines. Fortunately, publicly-available, anonymous health records help researchers rapidly identify and fight dangerous genetic variants.

But the existence of these well-intentioned public records turns Kucharski's narrative to a different epidemic: assaults on privacy. Kucharski recounts the classic "outing" of former Massachusetts Governor William Weld's health records, which were extracted from supposedly anonymous hospital records. This assault utilized a simple "genetic code"—publicly available voter records and Weld's name, age, and gender—to identify him. The resulting publicity eventually inspired significant changes to the way in which the U.S. stores and shares health-related data.[2] Kucharski offers additional disturbing examples about the many cracks in the shields that supposedly protect personal data.

The book's closing chapter, "A Spot of Trouble," provides an insider's prescient closing riff on what most newspaper readers now know about pandemics, misinformation, and other problems. Kucharski specifically warns that "the biggest challenges are often practical rather than computational," and adds that the messy, complicated nature of datasets reflect the human lives on which they are based.

[2] Not to mention fruitful research into differential privacy, among other forms of protection. See [1] for more details.

The Rules of Contagion is relevant to anyone who is interested in the roles of modeling and science in general; mathematical and biological barriers are nearly nonexistent. In fact, early chapters could perfectly frame an upper-level undergraduate or beginning graduate seminar. Participants could pair the text's descriptions of an epidemiological problem with the corresponding mathematics and modeling in papers from the complete bibliography (as could any reader seeking more technical detail). Readers could also draw on the closing chapter's daunting ethical dilemmas to approach a more complicated objective of broad scientific education: preparing students to fulfill the social obligations conferred by the gifts of scientific skill.

No one owns the rules of contagion. They are part of the shared grounds of scientific understanding and public good, and they arise in surprising settings. As a practitioner of the arts that he describes, Kucharski is a lively, intelligent, and well-informed guide to the data-heavy side of contagion modeling. His tour is alternately entertaining, gripping, and alarming, especially in this time of pandemic fear, misinformation, and crumbling privacy protections.

The Rules of Contagion provides an intellectual adventure ride and moral challenge; an account of scientific accomplishment; a list of daunting, as yet unanswered questions; and a narrative of battles against diseases won, lost, and in flux. Few books of its type are successful on so many fronts.

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A Moving Argument

MATHEMATICAL CURIOSITIES
By Mark Levi

Here is a twist on a well-known problem in mechanics. A cube rests on a sphere, as in Figure 1. The contact is of no-slip kind. What condition on the sizes h and r guarantees stability of the equilibrium?

A Solution by Motion

The standard solution involves expressing the cube's potential energy V as a function of the tilt angle θ and expressing the minimality condition $V''(0) > 0$ in terms of a and r. Although this is straightforward, it involves some calculation and is not very instructive. Instead, here is a solution with no calculation. If we roll the cube to the right without sliding (as in Figure 2), both the center of mass and the contact point with the sphere move to the right. The solution then amounts to the observation that *the equilibrium is stable if the horizontal velocities satisfy* $v_C > v_{\text{c.m.}}$ *at the moment the equilibrium is passed.*

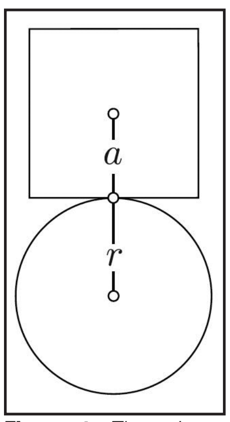

Figure 1. *There is no slip at the contact point.*

To translate this criterion into the condition on r and a, let ω be the cube's angular velocity; ω is thus also the angular velocity at which contact C travels around the circle. We therefore have

$$v_{\text{c.m.}} = \omega a, \quad v_C = \omega r.$$

The former is valid at the moment the equilibrium is passed since C is the instantaneous center of rotation and a is the distance of the center of mass to C. Substituting these values into our stability criterion $v_C > v_{\text{c.m.}}$ gives

$$\boxed{r > a.}$$

In other words, the equilibrium is stable if and only if the square does not hang over the sides of the circle.

Equivalence of $v_C > v_{\text{c.m.}}$ With the Minimality of $V(0)$

According to Figure 2b, the velocity $\mathbf{v}_{O'} \perp C'O'$ and $\mathbf{v}_{O'}$ is hence sloped upwards so that the potential energy is increasing.

General Shapes

The same idea applies to a general situation of a rock that is resting on a stationary rock (see Figure 3). As we roll the rock, the velocity of the center of mass is horizontal at the moment the equilibrium is passed;[1] the stability criterion is thus $v_C \cos\theta > v_{\text{c.m.}}$. To transform this into a geometrical condition, we note that $v_{\text{c.m.}} = \omega h$ and

$$v_C = \frac{\omega}{k_1 + k_2}, \qquad (1)$$

where k_1, k_2 are the curvatures of the two rocks (positive for convex rocks). Indeed, Figure 3b depicts an infinitesimal segment of the rock that has rolled from position AB to a new position $A'B'$ in time dt and has rotated (in addition to translation) through the angle ωdt. Therefore,

$$\angle(b, b') = \omega dt$$

(we count clockwise rotation as positive to avoid dealing with negatives). On the other hand, again treating all angles as positive,

$$\angle(b, b') = \angle(a, b) + \angle(a, b') = k_2 ds + k_1 ds.$$

Comparing the last two expressions for $\angle(b, b')$ yields $\omega dt = (k_1 + k_2) ds = (k_1 + k_2) v_C dt$, which amounts to (1).

We conclude that the rock in Figure 3 is stable iff $v_C \cos\theta > v_{\text{c.m.}}$, i.e., if

$$\boxed{\frac{\cos\theta}{k_1 + k_2} > h.} \qquad (2)$$

For the cube on the sphere in Figure 1, $k_1 = 1/r$ and $k_2 = 0$, $h = a$, $\theta = 0$, and (2) agrees with the result $r > a$.

[1] This is true because the velocity of O is orthogonal to the line OC from the instantaneous center of rotation C, and because OC is vertical at the moment in question.

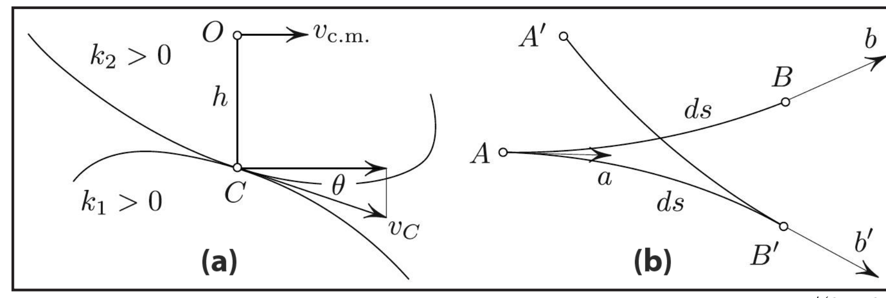

Figure 3. *Rolling the rock.* **3a.** *Stability criterion for a rolling rock:* $v_C \cos\theta > v_{\text{c.m.}}$. **3b.** $v_C = \omega / (k_1 + k_2)$.

Stability of Tilted Cubes

A cube can rest in equilibrium on any point of the sphere with $|\theta| < \pi/4$ (see Figure 4a); for the cube to be in equilibrium at a given θ, the point of contact must be at the distance $h\tan\theta$ from the midpoint of the side.

Which (if any) of these equilibria are stable? The answer is given by (2). We have $k_1 = 1/r$, $k_2 = 0$, and $h = a/\cos\theta$; the tilted equilibrium in Figure 4 is thus stable precisely if

$$a < r\cos^2\theta. \qquad (3)$$

For θ just under $\pi/4$, the largest stable cube will be just under $a = r/2$.

A Geometrical Criterion

Condition (3) looks nicer when expressed geometrically: *the equilibrium is stable iff the cube's center lies below the arc of the ellipse with the semiaxes* r *and* $2r$ (see Figure 4b). If we extend the cube's base (by a weightless extension) as in Figure 4b, all points on the upper semicircle—not only the ones with $|\theta| < \pi/4$—can serve as equilibrium contact points.

The counterintuitive stability of the small cube in Figure 4b contrasts with the instability of an inverted pendulum. The difference is that the "pivot"—i.e., the contact point—for the cube responds to perturbations so as to create a restoring torque: an inanimate stabilizing mechanism.

The figures in this article were provided by the author.

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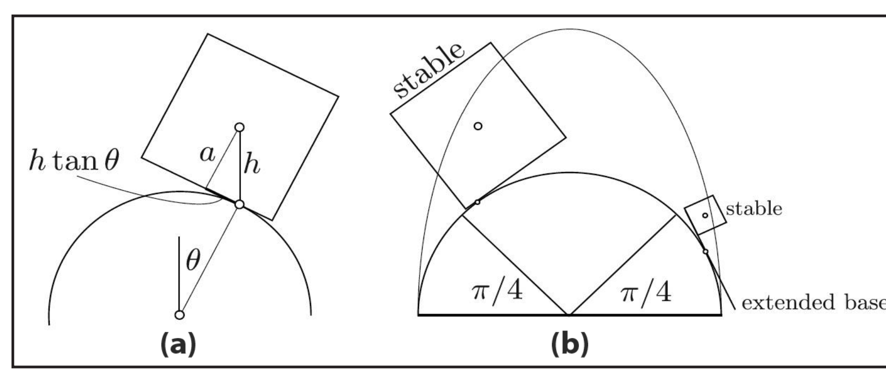

Figure 4. *Stability of tilted cubes.* **4a.** *A cube can be in equilibrium at any angle* $|\theta| < \pi/4$. **4b.** *The equilibrium is stable iff the cube's center lies below the ellipse with semiaxes* r *and* $2r$.

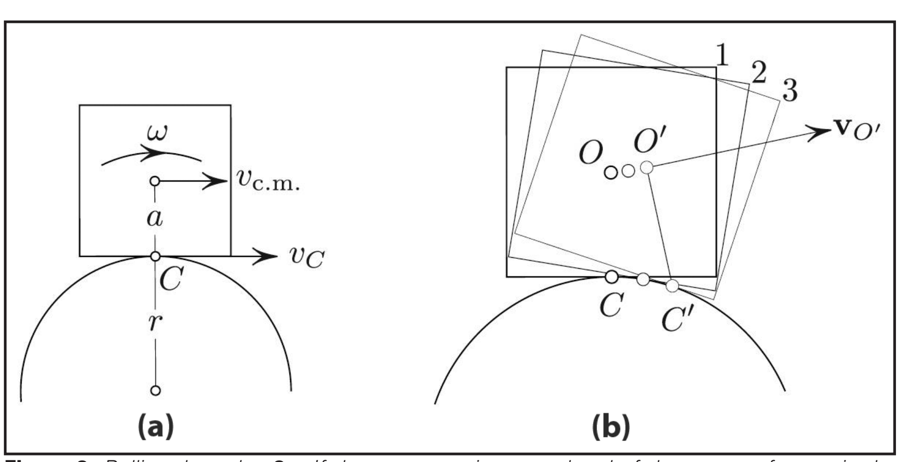

Figure 2. *Rolling the cube.* **2a.** *If the contact point gets ahead of the center of mass in the horizontal direction, the gravitational torque is then restoring towards the equilibrium.* **2b.** *More compactly, stability criterion is* $v_C > v_{\text{c.m.}}$.

BLIS

Continued from page 8

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Alicia Dickenstein Receives the L'Oréal-UNESCO For Women in Science International Award

Each year, the United Nations Educational, Scientific, and Cultural Organization (UNESCO) and the L'Oréal Foundation honor five exceptional female scientists with the L'Oréal-UNESCO For Women in Science International Awards.[1] Recipients are recognized for their scientific accomplishments, unique career paths, outstanding talents, and profound commitment to their professions in traditionally male-dominated fields. The award committee selects one recipient from each of five different regions: Africa and the Arab States, Asia and the Pacific, Europe, Latin America and the Carribb-ean, and North America. SIAM is happy to share that Alicia Dickenstein (University of Buenos Aires) is the 2021 laureate for the Latin America and Caribbean region!

Alicia Dickenstein of the University of Buenos Aires is also a member of the SIAM Council.

Per the prize citation, Dickenstein was "recognized for her outstanding contributions at the forefront of mathematical innovation by leveraging algebraic geometry in the field of molecular biology. Her research enables scientists to understand the structures and behavior of cells and molecules, even on a microscopic scale. Operating at the frontier between pure and applied mathematics, she has forged important links to physics and chemistry and enabled biologists to gain an in-depth structural understanding of biochemical reactions and enzymatic networks."

Dickenstein, who received her Ph.D. from the University of Buenos Aires, was recently elected to the SIAM Council and began her three-year tenure on January 1, 2021. She is also a member of the SIAM Activity Group on Algebraic Geometry and the SIAM Activity Group on the Life Sciences, and she serves as a corresponding editor for the *SIAM Journal on Applied Algebra and Geometry*. Dickenstein's research interests center on the computational aspects and applications of algebraic geometry, particularly in the context of polynomial and biological systems.

"My first reaction to receiving one of the five 2021 L'Oréal-UNESCO For Women in Science International Awards was, of course, great happiness on a personal level and gratitude to my family, students, and coauthors," Dickenstein said. "I very much hope that all the publicity around this award helps girls realize that mathematics is a career that everyone, regardless of gender (as well as race, social status, etc.), can choose and enjoy. I am also happy because math is not a very 'popular' discipline, so this is a recognition for all of us."

[1] https://en.unesco.org/science-sustainable-future/women-in-science/laureates

Behavioral Scoring: Markov Chains and Markov Decision Processes

By David Edelman and Jonathan Crook

The following is a short reflection from two of the authors of Credit Scoring and Its Applications (Second Edition), which was published by SIAM in 2017 and written by Lyn Thomas, Jonathan Crook, and David Edelman. The updated, more robust second installment is a follow-up to the first edition, which was released in 2002.

This text serves as an analysis of section 7.4, entitled "Behavioral Scoring: Orthodox Markov Chain Approach." Portions of this summary have come directly from the book and were modified slightly for clarity.

Researchers first suggested the idea of modeling consumers' repayment and usage behavior in the early 1960s. They sought to identify the different possible states of a borrower's account and estimate the chance of the account moving from one state to another between billing periods. These states are primarily contingent upon the account's current position and recent history, but can also depend on the initial application. Therefore, typical information that one could use to determine an account's state might include its current balance, the number of overdue time periods, and the number of "reminder" letters in the last six months. The object is to define states in such a way that the probability of moving to any particular state at the next billing period is dependent only on the account's current state; this is the definition of a Markov chain.

In 1983, Jarl Kallberg and Anthony Saunders employed a simple version of this model [5]. The data in their example led to a stationary transaction matrix (see Figure 1).

Thus, if all of the accounts have no credit—i.e., $\pi_0 = (1,0,0,0,0)$—at the beginning, the account distribution is $\pi_1 = (0.79, 0.21, 0, 0, 0)$ after one period. After subsequent periods, it becomes

$$\pi_2 = (0.64, 0.32, 0.04, 0, 0),$$

$$\pi_4 = (0.468, 0.431, 0.070, 0.023, 0.008),$$

$$\pi_{10} = (0.315, 0.512, 0.091, 0.036, 0.046).$$

This is a useful way to estimate the amount of bad debt that will appear in future periods. After 10 periods, the model estimates that 4.6 percent of the accounts will be bad.

One of the first Markov chain models of consumer repayment behavior was a credit card model suggested by Richard Cyert, H. Justin Davidson, and Gerald Thompson [2], wherein each dollar owed would jump from state to state. This model had trouble accounting for conventions and allocating part payments; a different group later addressed these concerns [8]. A. Wayne Corcoran suggested another variant of the model that utilized different transition matrices for accounts with different sized loans [1], and David Edelman proposed a version with different transition matrices for different months of the year [4]. Kallberg and Saunders also investigated models for which the state space depends on the amount of the opening balance and the level of repayments [5].

In more sophisticated Markov chain models, each state s in the state space has three components: $s = (b, n, i)$. Here, b is the outstanding balance, n signifies the number of current consecutive periods of nonpayment, and i represents the other characteristics of importance. One can use the anticipated one-period reward that the lender makes from a customer in each state to calculate the expected total profit from that customer under any credit limit policy. In fact, it is possible to calculate the credit limit policy that maximizes the profits for a given level of bad debt — or minimizes the bad debt for a given level of profit.

We employ dynamic programming to calculate the chances of defaulting, $D(b,n,i)$, for a simple example in a mail-order context. Utilizing past data, we develop a table of default probabilities for each state and the expected value of the state's orders (see Figure 2). The states are ordered by increasing probability of default, where the total expected order value is £1,000. The lenders must then decide on an acceptable default level and value of lost orders. Management has to thus choose which default level D^* is acceptable. Since $D(b,n,i)$ is increasing in b, one can solve $D(L(n,i),n,i) = D^*$ to find the credit limit $L(n,i)$ for each state (n,i).

This Markov chain model has two elements. The first element is the states, or the consumers' repayment performance; a transition matrix describes the repayment behavior's dynamics. The second is a reward function: the value of the consumer's state to the organization. The addition of a third element—potential decisions of the lender that impact the rewards and transitions between states—yields a Markov decision process, which is an example of stochastic dynamic programming.

We can extend the previous example to solve for the optimal credit limit policy, which serves as a function of the state s and the number of periods until the end of the time horizon.

More recent Markov decision models that maximize the credit limit policy employ other state descriptions. One research group—which received the Daniel H. Wagner Prize for Excellence in the Practice of Advanced Analytics and Operations Research—utilized states with six components, such as default rate, usage, purchase behavior, and repayments behavior and amount [7]. Each component was split into two, three, or four bands for a total of nearly 600 behavioral states. Actions in this model included setting the credit limit (10 levels) and deciding what interest rate to charge (five levels).

Meko So and Lyn Thomas allowed 10 credit limits and used behavioral score bands—along with "defaulted," "closed," and "inactive"—as their model's eight states [6]. They found that a second-order Markov chain that defines the state based on the current and previous behavioral score bands provides a better fit to the data. They also introduced a method that estimates the transition probabilities from states with minimal data.

One can apply the Markov decision approach to optimize the collection process for defaulted loans. In this case, the lender seeks to recover as much of the defaulted debt as possible; the amount recovered as a fraction of the original defaulted debt is called the recovery rate. Lenders can undertake a number of actions with varying levels of harshness and cost implications, ranging from gentle reminder telephone calls to legal action. They must also decide the length of time for which they will undertake an action and plan the actions' sequence. The defaulters' state is described by a lender's current collection action, the ongoing duration of that action, and the recovery rate that was obtained from the borrower when the current action began. Researchers used this type of Markov decision process model to find the sequence and duration of actions that maximize the expected recovery rate [3].

From/To	NC	0	1	2	3
NC	0.79	0.21	0	0	0
0	0.09	0.73	0.18	0	0
1	0.09	0.51	0	0.40	0
2	0.09	0.38	0	0	0.55
3	0.06	0.32	0	0	0.62

Figure 1. *A stationary transition matrix based on the model by Jarl Kallberg and Anthony Saunders.*

State	s_7	s_1	s_3	s_2	s_9	s_8	s_4	s_6	s_5
Default probability	.001	.004	.008	.02	.06	.09	.12	.15	.2
Value of orders (£)	50	150	250	200	120	80	60	40	50

Figure 2. *Table of default probabilities for each state, along with the expected value of the state's orders.*

FROM THE SIAM BOOKSHELF

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Enjoy this passage? Visit the SIAM bookstore[1] to learn more about Credit Scoring and Its Applications[2] and browse other SIAM titles.

David Edelman, who holds degrees in statistics from the University of Glasgow and Harvard University, embarked on an academic career before moving into banking. He then spent 17 years working in credit risk management for two major Scottish banks. In 2003, Edelman began operating as a freelance consultant, interim manager, and trainer. In the past 17 years, he has served hundreds of clients from over 30 countries and across five continents. Jonathan Crook is Professor of Business Economics and Director of the Credit Research Centre at the University of Edinburgh's Business School. His research concentrates on statistical models to predict credit risk. Crook has published almost 70 journal articles, mostly on credit scoring, and has co-authored or co-edited five books. He has also acted as a credit scoring consultant for a number of banks.

1 https://my.siam.org/Store

2 https://my.siam.org/Store/Product/viewproduct/?ProductId=28779436

SIAM Undergraduate Research Online Prepares Students for Career Publishing Endeavors

By Joanna Wares and Luis Melara

For students studying science, technology, engineering, and mathematics (STEM), the rich realm of undergraduate research serves as a high-impact learning environment that transports them beyond the classroom and allows them to make original contributions to their disciplines. Research projects—during which students work on problems that align with their personal interests and possible career aspirations—have thus become essential components of many undergraduate programs. Applied and industrial mathematics undergraduates who partake in research learn to apply proper mathematical and computational techniques, hone their communication skills, write journalistic articles, and summarize methods and results. These writing tasks are also valuable learning experiences that help prepare students for higher education or postgraduate positions. *SIAM Undergraduate Research Online* (SIURO)[1] is a web-based publication that encourages students to undergo this process by offering them a space to showcase their work.

SIURO was established in 2008 to publish high-quality undergraduate research in applied and computational mathematics. The publication covers a wide array of topics, including analysis, discrete mathematics, operations research, optimization, statistics, dynamical systems, modeling, and general computation. Student authors employ these methods in applications that pertain to a variety of fields, ranging from the physical and life sciences to finance and management.

One of SIURO's primary objectives is to provide undergraduates with a platform where they can experience peer review procedures firsthand. Throughout the publication process, students write and submit manuscripts, serve as corresponding authors, generate responses to review reports, and communicate directly with members of the SIURO Editorial Board (industry experts and international faculty from top universities and liberal arts colleges with expertise in the publication's broad-ranging fields). Faculty members, graduate students in Ph.D. tracks, or industry leaders review all submissions and produce referee reports, which provide important feedback. Student authors may also receive review reports that request minor and/or major revisions; they can choose to address and respond to these amendments and resubmit their work for reconsideration, as is typical with peer-reviewed publications.

[1] https://www.siam.org/publications/siuro

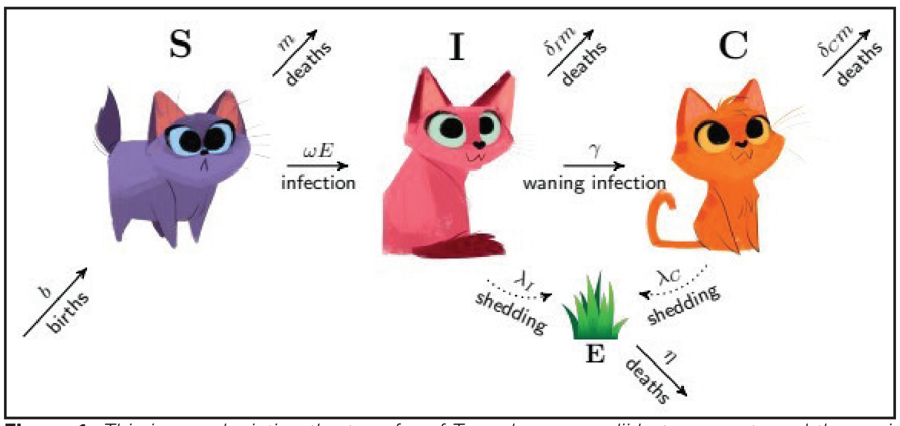

Figure 1. *This image depicting the transfer of Toxoplasma gondii between cats and the environment was featured in a 2018 SIAM Undergraduate Research Online (SIURO) article by Emily Kelting. Figure courtesy of [1].*

In recent years, disease modeling has been a particularly popular research area for students. An example of a successful article is Emily Kelting's 2018 paper about *Toxoplasma gondii* in cats, which can also affect pregnant women [1]. This work describes conditions that minimize the risks of parasite transfer from cats to other species (see Figure 1). Unsurprisingly, COVID-19 was a popular research area in 2020, and SIURO authors approached the topic with various techniques. All SIURO papers are open access and available online.

SIURO will be undergoing several changes to its publication policies in 2021. Beginning in the spring of this year, all research mentors will now be listed as "project advisors" on published papers; proper citations of contributors will thus include project advisors as well as the undergraduate authors. In addition, Joanna Wares (University of Richmond) was appointed editor-in-chief of SIURO in January after serving as an associate editor for five years. She succeeds Luis Melara (Shippensburg University), who led the publication for the last six years. During his time as editor-in-chief, Melara systematized and simplified SIURO's submission system to improve ease of use for submitters and reviewers. He even developed a system and criteria for review that ensures the accomplishment of SIURO's goals: encouraging undergraduate research and providing an outlet for meritorious study.

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Joanna Wares is an associate professor of mathematics at the University of Richmond and the current editor-in-chief of SIAM Undergraduate Research Online (SIURO). Her research interests include best practices for intertwining social justice with mathematics, as well as modeling in the medical sciences and public health. Luis Melara is an associate professor of mathematics and assistant director of the Wood Honors College at Shippensburg University. His research focuses on mathematical biology and optimal control. Melara is also interested in increasing the number of traditionally underrepresented groups in the mathematical sciences.

Mathematics in Industry

Continued from page 7

healthcare (see Figure 1, on page 7). Taylor had previously worked in industry for several years after obtaining his bachelor's and master's degrees, before opting to pursue a Ph.D. in mechanical engineering at Stanford. His Ph.D. thesis focused on the integration of computational fluid dynamics (CFD) with medical imaging modalities to predict blood flow and pressure in patient-derived geometries. After becoming a professor at Stanford, Taylor saw a chance to bring his ideas to the public through the healthcare field. "I went into academia because there was an opportunity to work on a hard problem for a long time, which a lot of industries weren't ready for yet," Taylor said. "I decided to start HeartFlow because I was at a point where I could pursue the ideas from my Ph.D. on a larger scale."

HeartFlow has since provided a new FDA-cleared tool for diagnosing and managing coronary artery disease, all driven by Taylor's initial work in cardiovascular CFD and the growing field of machine learning (see Figure 2). "Use your thesis as a starting point for your career," Taylor said. "Ask yourself three things: What do you love? What are you good at? And what does the world need? Know these three things and always be flexible; what you start doing after your degree will change over the years."

Which Should I Choose?

The choice between a career in academia or BIG ultimately depends on individual preference, research area, and desired lifestyle. Given the complexity of the decision, SIAM has taken numerous steps to better educate undergraduate and graduate students on the nuances of BIG careers. For example, mathematical sciences students and faculty can utilize SIAM's Visiting Lecturer Program,[3] which provides the community with a roster of experienced mathematicians and computational scientists from both academia and BIG that are willing to speak to students about their experiences. The Tondeur Initiatives,[4] funded by Philippe and Claire-Lise Tondeur in 2018, also provide a repository of programming that aligns with the BIG Math Network[5] and pertains to BIG internships and career opportunities.

One obvious limitation of academia is the job market itself. "The number of tenure-track positions remains fairly flat, while the number of other types of positions is exploding," Levy said. While this remark is true in typical years, the COVID-19 pandemic has further complicated the situation. In fact, a recent article in *Science* reports a 70 percent decrease in STEM tenure-track openings in the U.S. when compared to the previous year [1]. Taylor suggests that new applied math Ph.D.s consider possible academic employment in departments beyond mathematics. "Interdisciplinary scientists can be qualified for different department positions," he said. For instance, experts in uncertainty quantification may find suitable placement in mechanical engineering departments, whereas those who excel at mathematical biology might be ideal candidates for biostatistics or developmental biology departments. The same logic applies to industry positions, where the role of "data scientist" takes many different forms depending on whether one is employed at a national laboratory or social media company.

Ellwein Fix encourages early-career mathematicians to use conferences—including virtual gatherings—as opportunities to learn more about academic versus BIG careers. "Don't hesitate to contact the people you meet at conferences," she said. "Senior researchers and professors expect emails from students who ask to meet during conferences, so be sure to reach out to those who are making an impact in your field of study." Ellwein Fix still communicates with multiple BIG contacts that she encountered at prior conferences; they often inform her of new industry jobs for her own students. "Many of my career opportunities came from staying in touch with individuals I met during my Ph.D.," she said. Some conferences also host career fairs—the 2021 SIAM Conference on Computational Science and Engineering[6] is one such example—which are great ways for students to learn about industry jobs at different companies. Representatives from these companies can also answer questions about work-life balance, job growth, and benefits.

Taylor likens the entry-level, assistant professor position in academia with having a startup company. "As an assistant professor, you have numerous new responsibilities to which you must adapt, all while trying to obtain your own funding," he said. This is especially true at Research 1 institutions. Industry positions are a bit different. "You will likely have to stay in your own lane in an industry job at a big company," Taylor continued. "It can be easy to get stuck in the same role, so it is important to have a job at a company where you know you can grow." Because both academic and industrial or government jobs have their own pros and cons, new graduates should assess their individual goals inside and outside of their professions.

Many resources are available for current and recent graduate students who are deciding between careers in industry and academia. The *BIG Jobs Guide* and BIG Math Network provide information about opportunities outside of the realm of academia, in addition to resources about BIG careers. Additional information is accessible via SIAM's Career Resources page.[7]

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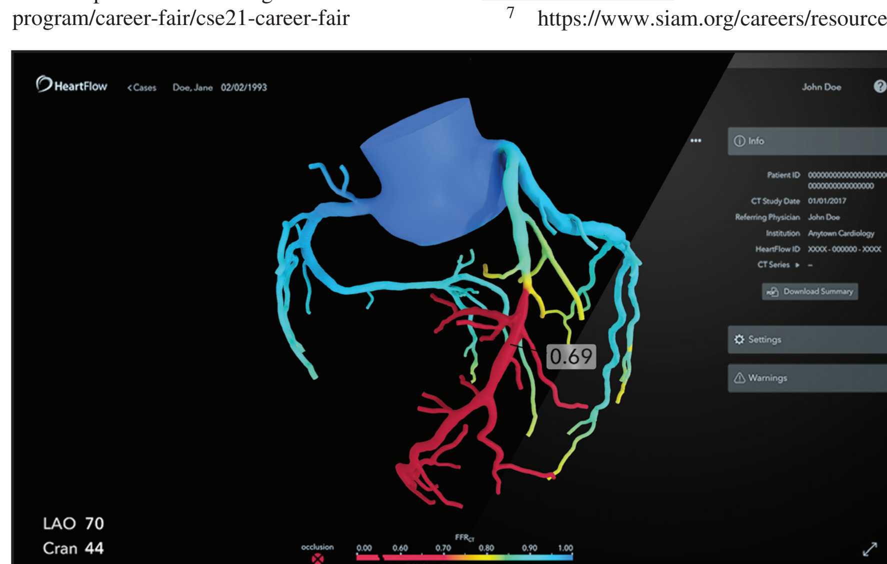

Figure 2. *The HeartFlow Analysis interactive three-dimensional model enables physicians to precisely diagnose and manage coronary artery disease. Figure courtesy of HeartFlow.*