Linear extensions of a partial order subject to algebraic constraints

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Abstract

We present a novel combinatorial problem that arises from mathematical biology. In order to understand the dynamics of models of gene regulatory networks over a parameter space, a problem of constructing linear extensions of a partial order with algebraic constraints arises naturally. We formulate the problem for a class of algebraic constraints related to the form of nonlinearities in the gene regulation model. We provide an algorithm that partially solves the problem. We formulate a conjecture on the special role of additive constraints in the class of all considered constraints. We present several examples where we show that the number of solutions is much smaller than the number of unconstrained linear extensions.

1 Introduction

In this paper we formulate and study a problem of counting linear extensions of a fixed partial order that are subject to additional algebraic constraints. We propose a framework for a solution to the problem and state and prove several key results in this framework. Our motivation comes from the study of dynamics of regulatory networks over parameter space.

The computation of linear extensions from partial orders is a well-known problem [1, 2]. We consider an extension of this problem in which there is conditional dependence between the inequalities in the partial order. In other words, the choice of one inequality can determine the choice of another inequality when constructing a linear extension of the partial order.

In our case, we consider a conditional dependence that is described by an algebraic expression M in k variables σ_n, written as a vector $\sigma = (\sigma_n)$. Each $\sigma_n \in \{a_n, b_n\}$ can take on two constant values, a_n and b_n with $a_n < b_n$. The elements of the partial order are then all of the realizations $M \circ \sigma$ over the set of all combinations of values of $\sigma_n,\ n = 1, \dots, k$, which is a collection of 2^k values. The collection of constraints $a_n < b_n,\ n = 1, \dots, k$, imposes a partial order on the 2^k elements that has the structure of a hypercube. However, the algebraic form of M imposes additional constraints. For example, if $M = \sigma_1 + \sigma_2 + \sigma_3$ and during the construction of a linear order we choose $a_1 + b_2 + a_3 < a_1 + a_2 + b_3$, then by necessity $b_1 + b_2 + a_3 < b_1 + a_2 + b_3$. This is the conditional dependence referred to earlier.

We present a partial solution to this problem: an algorithm that requires an oracle for a decision question. Although we cannot provide an algorithm for the oracle, we present a method for counting all linear extensions of a constrained partial order for a multilinear M, given responses from the oracle.

We begin by defining regulatory networks as a motivation for our work. We show how a partial order arises on elements that are themselves algebraic expressions in the parameters of

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the dynamical system associated to the regulatory network. In Section 4, we introduce an oracle and an algorithm that queries the oracle in order to construct the algebraically constrained linear extensions, and in Section 5 we discuss stronger results for a class of partial orders where M is an additive function. We conclude with examples demonstrating our results.

2 Switching systems

In this section, we introduce a particular class of ordinary differential equations (ODEs) as motivation for the origin of partial orders with conditional dependence. These ODEs describe the dynamics of the concentrations of interacting molecular species. The interactions between molecular species are compactly described by a regulatory network. Typically, a regulatory network is represented as an annotated directed graph, where each vertex represents the concentration of a molecular species (e.g. protein, mRNA), and an edge from node i to node j indicates that i directly regulates j, as in [5].

Definition 2.1. A *regulatory network* **RN** is a finite directed annotated graph (V, E) with vertices $V = \{1, \dots, N\}$ and edges $E \subset V \times V \times \{\rightarrow, \dashv\}$. An edge annotated $\rightarrow$ indicates activation, while $\dashv$ indicates repression. By convention we write $i \rightarrow j$ or $i \dashv j$ if $(i, j, \rightarrow) \in E$ or $(i, j, \dashv) \in E$, respectively. We say i *regulates* j if $i \rightarrow j$ or $i \dashv j$. Self-edges may exist for some nodes, but for each pair $(i, j) \in V \times V$, there exists at most one edge from i to j.

One of the common models for the dynamics of these systems uses a system of coupled ordinary differential equations, where each equation describes the evolution of a single species. It is typically assumed that the interactions have switch-like behavior and thus the nonlinearities in these ODEs are often assumed to have a sigmoidal shape. One common approximation [3, 4, 6, 7, 8, 9, 10, 11, 12, 13] yields nonlinearities that take the form of piecewise constant functions.

Definition 2.2. Given a regulatory network, the associated *switching system* takes the following form

$$\dot{x}_j = -\gamma_j x_j + f_j(x), \quad j = 1, \dots, N \tag{1}$$

where N is the number of nodes in the regulatory network and for all j, $\gamma_j > 0$. Consider a particular j and let $S(j) = \{i \mid y_i \text{ regulates } y_j\}$ and $k = |S(j)|$, and for each $i \in S(j)$ let $\theta_{j,i}$ be a threshold value associated to the edge from i to j (recall from Definition 2.1 there is a unique such edge). This threshold is a value about which i behaves as an ON/OFF switch for j as in Figure 1.

We define $f_j = M_j \circ \sigma_j$ where $M_j : \mathbb{R}^k \to \mathbb{R}$ is a *logic function* (which we explain in greater detail below) and $\sigma_j : D \to \mathbb{R}^k$, $D \subset \mathbb{R}^N$, is a multidimensional step function, which we define by

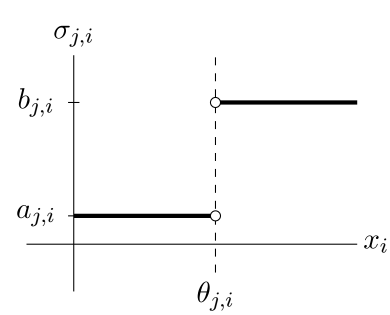


Figure 1: An example of an ON/OFF switch for j described in Definition 2.2.

its coordinate projections $\sigma_{j,i} = \pi_i \circ \sigma_j$ (whenever $i \in S(j)$):

$$\sigma_{j,i}(x) = \begin{cases} a_{j,i} & \text{if } i \to j \text{ and } x_i < \theta_{j,i} \text{ or } i \dashv j \text{ and } x_i > \theta_{j,i} \\ b_{j,i} & \text{if } i \to j \text{ and } x_i > \theta_{j,i} \text{ or } i \dashv j \text{ and } x_i < \theta_{j,i} \end{cases} \tag{2}$$

where the inequalities are evaluated at the point $x = (x_1, \ldots, x_N)$ in the domain D, and for $j = 1, \ldots, N$ and each $i \in S(j)$, $0 < a_{j,i} < b_{j,i}$ (note the assumption $0 < a_{j,i}, b_{j,i}$ is reasonable for biological applications). D is defined to be

$$D = \left\{(x_1, \ldots, x_N) \in \mathbb{R}^N \;\middle|\; x_i \neq \theta_{j,i} \text{ for any } i, j\right\}$$

M_j is called a logic function because it determines the type of regulatory interactions via the logical operators AND ($\wedge$) and OR ($\vee$) acting on Boolean values. For example, suppose a Boolean variable $B_j = 1$ only if $(B_i = 1) \vee (B_k = 1)$. Then $B_j = \min(1, B_i + B_k)$. On the other hand, suppose $B_j = 1$ only if $(B_i = 1) \wedge (B_k = 1)$; then $B_j = B_i * B_k$. One can see that there is a natural relationship between addition and OR, and one between multiplication and AND. In our regulatory network, the condition "j is regulated if $i \vee k$ pass threshold" translates into an additive function so that

$$\dot{x}_j = -\gamma_j x_j + \sigma_{j,i}(x) + \sigma_{j,k}(x)$$

while likewise "j is regulated if $i \wedge k$ pass threshold" becomes

$$\dot{x}_j = -\gamma_j x_j + \sigma_{j,i}(x)\sigma_{j,k}(x).$$

In the first case, $M_j(p, q) = p + q$ and in the second $M_j(p, q) = pq$. For more than two regulators, we consider logical expressions in which each regulator appears exactly once, and we replace $\vee$ with $+$ and $\wedge$ with $*$ to arrive at M_j. This class of functions is called *multilinear*. It is readily verified that for $0 < a_i, b_i$, the range of $M_j \circ \sigma_j$ is a subset of $\mathbb{R}^+$.

The thresholds $\theta_{j,i}$ play an important role in the dynamics of (1). In particular, these thresholds divide phase space into finitely many domains with the property that flow is mono-directional across each face of each domain. Using this domain structure, Cummins et al. [5] have recently developed a method to describe the coarse dynamics of the system (1) over the parameter space. This description of the system's dynamics centers on an object called the *parameter graph*, a finite undirected graph which discretizes parameter space into a finite set of *parameter nodes*; each of these is a set of parameters that all give rise to the same direction of flow across each domain's boundary in phase space, and consequentially, the same coarse dynamics.

For our purposes it will be sufficient to consider a single node j and the associated function f_j. Because of this, the nodes regulating j are arbitrary, and accordingly we can assume $S(j) = \{1, \ldots, k\}$ without loss of generality. We will simplify notation by dropping the j index, so that instead of writing $f_j = M_j \circ \sigma_j$, we write $f = M \circ \sigma$, and each element of σ is now written with a single index, σ_i.

3 Partial orders with algebraic constraints

We can now address the main focus of this paper; roughly speaking, we deal with the outputs of f, on which there exists a partial order. We seek to count the linear extensions of this partially ordered set that respect algebraic constraints imposed by the logic M and the arithmetic of $\mathbb{R}$. This tells us how many ways the parameters of the system can be arranged. We make these notions rigorous here.

Definition 3.1. We define the finite discrete set

$$\mathcal{F} := \left\{ M(c_1, \dots, c_k) \in \mathbb{R}^+ \;\middle|\; c_i \in \{a_i, b_i\} \text{ for } i = 1, \dots, k \right\},$$

where recall that a_i and b_i are fixed parameters of the function σ_i, and M is the logic function associated to f.

Definition 3.2. A *partially ordered set* or *poset* $(U, \trianglelefteq)$ is a set U together with a reflexive, anti-symmetric, transitive binary relation $\trianglelefteq$ on U called a *partial order*. If in addition $x \trianglelefteq y$ or $y \trianglelefteq x$ for each $x, y \in U$, then $\trianglelefteq$ is a *linear order*.

Throughout this work, any partial order or linear order $\trianglelefteq$ will be denoted with a horizontal bar, indicating possible equality, unless the order is strict, in which case it will be omitted. For instance, $x \trianglelefteq y$ means $x \triangleleft y$ or $x = y$, while $x \triangleleft y$ implies $x \neq y$. Therefore, $\triangleleft$ is irreflexive.

A *linear extension* of a partial order $\triangleleft$ on a set U is a linear order $\trianglelefteq^*$ on U such that for any $x, y \in U$ satisfying $x \trianglelefteq y$, we have $x \trianglelefteq^* y$.

Let $(U, \trianglelefteq), (V, \trianglelefteq')$ be two posets where $V \subset U$ and $\trianglelefteq, \trianglelefteq'$ may each be either a partial or linear order. If it is never true that $x \triangleleft' y$ and $y \triangleleft x$ simultaneously for any $x, y \in V$, then we say that $\trianglelefteq'$ *respects* $\trianglelefteq$. Note that with the appropriate domain restriction, we may also say that $\trianglelefteq|_V$ respects $\trianglelefteq'$.

There exists a natural partial order on the set $\mathcal{F}$, which arises entirely from the assumption that $0 < a_i < b_i$ for $i = 1, \dots, k$.

Definition 3.3. We define a partial order $\leq$ on $\mathcal{F}$ in the following way. Let $c_i, d_i \in \{a_i, b_i\}$ for $i = 1, \dots, k$. Then $M(c_1, \dots, c_k) < M(d_1, \dots, d_k)$ if and only if there exists a nonempty subset $I \subset \{1, \dots, k\}$ such that $c_i = a_i$ and $d_i = b_i$ if $i \in I$, and $c_i = d_i$ otherwise.

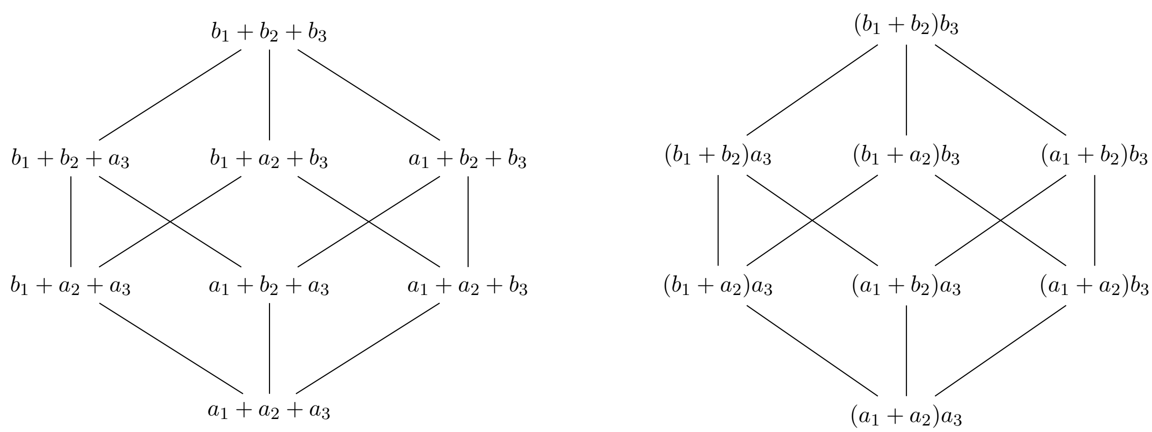
Remark 3.4. Note that in Definition 3.3 we abuse notation by using $\leq$ to denote the partial order on $\mathcal{F}$, while previous usage of this symbol (for instance, the assumption $0 < a_i < b_i$) has referred to the standard linear order on $\mathbb{R}$. This linear order on $\mathbb{R}$ and the partial order $\leq$ are not synonymous, and furthermore, even the restriction of the former to $\mathcal{F}$ is not equal to the partial order. However, it is easy to verify that if $M(c_1, \dots, c_k) < M(d_1, \dots, d_k)$ in the partial order, then $M(c_1, \dots, c_k) < M(d_1, \dots, d_k)$ in the linear order on $\mathbb{R}$. Because we deal with linear extensions $\leq^*$ of $\leq$, it will be important to keep in mind that not every $\leq^*$ respects the standard order on $\mathbb{R}$. See Definition 3.6.

Remark 3.5. A consequence of Definition 3.3 is that there is always a unique minimal element $M(a_1, \dots, a_k)$ and a unique maximal element $M(b_1, \dots, b_k)$ of $(\mathcal{F}, \leq)$.

We illustrate $(\mathcal{F}, \leq)$ visually by a Hasse diagram. For instance, Figure 2 displays two Hasse diagrams for $(\mathcal{F}, \leq)$ when $k = 3$, with differing logic M. Note the partial order is independent of M.

Definition 3.6. We denote the set of all linear extensions of the partial order $\leq$ by W. Further, we say a linear extension $\leq^*$ of $\leq$ is *realizable* if there exists a particular tuple $(a_1^*, b_1^*, \dots, a_k^*, b_k^*) \in \mathbb{R}^{2k}$, with $0 < a_i^* < b_i^*$ for $i = 1, \dots, k$, such that $\leq^*$ equals the restriction of the usual order on $\mathbb{R}$ to $\{M(c_1, \dots, c_k) \mid c_i \in \{a_i^*, b_i^*\} \text{ for } i = 1, \dots, k\}$. We denote the set of all realizable linear extensions of $\leq$ by Υ. If $\mathcal{S} \subset \mathcal{F}$, then we say a linear order $\trianglelefteq$ on $\mathcal{S}$ is realizable if there exists a realizable linear extension of $\leq$ whose restriction to $\mathcal{S}$ equals $\trianglelefteq$.

A fundamental open problem in the study of parameter graphs of regulatory networks is explicit construction of Υ, or at least a computation of its magnitude $|\Upsilon|$. This seems to be a difficult problem in algebraic geometry, since it involves enumeration of semi-algebraic sets in the space



(a) Additive M.

(b) Additive-multiplicative M.

Figure 2: Partial order on $\mathcal{F}$ when $k = 3$.

of parameters that are determined by inequalities involving multilinear functions [5]. The sets W and Υ are not the same in general, since not every linear extension of $\leq$ satisfies the algebraic constraints from M. For instance, note the following (in which the minimal and maximal elements from Remark 3.5 are omitted), is by definition a linear extension of the partial order $\leq$ in Figure 2a.

$$b_1 + a_2 + a_3 <^* a_1 + b_2 + a_3 <^* a_1 + a_2 + b_3 <^* b_1 + a_2 + b_3 <^* b_1 + b_2 + a_3 <^* a_1 + b_2 + b_3.$$

Yet under the linear order $\leq$ on $\mathbb{R}$, $a_1 + b_2 + a_3 < a_1 + a_2 + b_3$ implies $(b_2 - a_2) < (b_3 - a_3)$ while $b_1 + a_2 + b_3 < b_1 + b_2 + a_3$ implies $(b_3 - a_3) < (b_2 - a_2)$. Hence $\leq^*$ is not realizable.

4 Compatibility and Block Meshing

In this section, we present the most general results that we have for multilinear expressions M. This includes the introduction of an oracle called Compatible, an algorithm for counting algebraically constrained linear extensions that depends on the oracle, and several counting theorems. We begin by explaining that there is a natural "block structure" of the partial order induced by f.

Definition 4.1. Let $\ell \in \{0, 1, \ldots, k\}$. Then the ℓ^{th} *block* is the set

$$\Delta_\ell := \{M(c_1, \ldots, c_k) \in \mathcal{F} \mid \ell = |\{c_i = b_i\}|\}.$$

In other words, the ℓ^{th} block is the subset of $\mathcal{F}$ whose elements are such that exactly ℓ values $\sigma_i(x) = b_i$, referring back to (2) and dropping the j index. See Figure 3.

Remark 4.2. We remark that the nature of our problem causes the cardinality of the blocks to follow the Pascal sequence in the k^{th} row of Pascal's triangle, where k is the number of elements in the algebraic expression. For example, in Figure 3 where there are three elements, the cardinality of the blocks follows the sequence 1 3 3 1.

Division into blocks Δ_ℓ, $\ell = 0, 1, \ldots, k$, has two advantages. The first of these arises when we make the following definition.

Definition 4.3. Let $\mathcal{P}(\mathcal{F})$ be the power set of $\mathcal{F}$. We define a partial order $\preceq$ on $\mathcal{P}(\mathcal{F})$ in the following way. If $U, V \subset \mathcal{F}$, then $U \preceq V$ if for every $u \in U$ there exists $v \in V$ such that $u \leq v$ in $(\mathcal{F}, \leq)$.

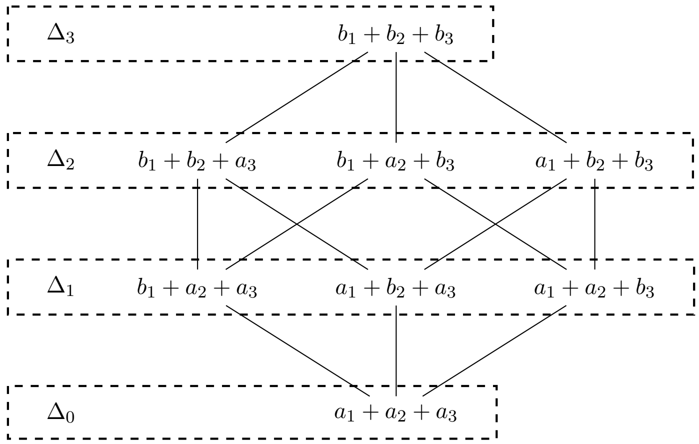


Figure 3: Blocks within $(\mathcal{F}, \leq)$ when $k = 3$ and M is additive.

The following is immediate from Definition 4.3.

Lemma 4.4. *The blocks Δ_ℓ, $\ell = 0, \ldots, k$, satisfy*

$$\Delta_0 \preceq \Delta_1 \preceq \ldots \preceq \Delta_{k-1} \preceq \Delta_k,$$

thus forming a linearly ordered subset of $\mathcal{P}(\mathcal{F})$.

The second advantage is that the elements within each block Δ_ℓ are unordered in $(\mathcal{F}, \leq)$. Therefore it is straightforward to count all possible linear orders of each block Δ_ℓ; their number is simply $|\Delta_\ell|!$.

This suggests the following approach to generating a linear order on $\mathcal{F}$. We first impose a linear order on each Δ_ℓ, and then merge them into a linear order on $\mathcal{F}$. There are two key difficulties in the last step. The first is that selecting a linear order on a particular Δ_ℓ may impose restrictions on the linear orders of Δ_i for $i \neq \ell$. We call this issue *compatibility* and define it in Definition 4.6. The second is that the linear orders on Δ_ℓ may support multiple realizable linear extensions of $\leq$, since they can be merged in multiple ways. We call this process *block meshing*, which we define in Definition 4.8.

Remark 4.5. Throughout the rest of this work we will refer to several arbitrary partial or linear orders on subsets of $\mathcal{F}$. Unless otherwise stated, if $U \subset \mathcal{F}$, then we will assume any partial or linear order on U to which we refer must respect the partial order $\leq$ on $\mathcal{F}$.

Definition 4.6. Let $U, V \subset \mathcal{F}$, $U \cap V = \emptyset$, and $U \preceq V$. Let $\trianglelefteq'$ and $\trianglelefteq''$ be a pair of linear orders on U and V, respectively. We denote the partially ordered set formed by the disjoint union of $\trianglelefteq'$ and $\trianglelefteq''$ by $(U \cup V, \trianglelefteq)$. Recalling Definition 3.6, we say that $\trianglelefteq'$ and $\trianglelefteq''$ are *compatible* if there exists a realizable linear extension $\trianglelefteq^*$ of $\trianglelefteq$.

As an example of compatibility, again consider the poset $(\mathcal{F}, \leq)$ in Figure 2a. If we impose linear orders $\trianglelefteq'$ and $\trianglelefteq''$ on Δ_1 and Δ_2, respectively, such that

$$b_1 + a_2 + a_3 \triangleleft' a_1 + b_2 + a_3 \triangleleft' a_1 + a_2 + b_3$$

and

$$b_1 + b_2 + a_3 \triangleleft'' b_1 + a_2 + b_3 \triangleleft'' a_1 + b_2 + b_3,$$

there exists a linear order $\trianglelefteq^*$ on $\Delta_1 \cup \Delta_2$ such that

$$b_1 + a_2 + a_3 \triangleleft^* a_1 + b_2 + a_3 \triangleleft^* a_1 + a_2 + b_3 \triangleleft^* b_1 + b_2 + a_3 \triangleleft^* b_1 + a_2 + b_3 \triangleleft^* a_1 + b_2 + b_3,$$

and $\trianglelefteq^*$ respects the partial order $\leq$ on $\mathcal{F}$ (see Figures 2a,3). Furthermore, this order is realized at the tuple $(a_1, b_1, a_2, b_2, a_3, b_3) = (1, 4, 2, 6, 3, 8)$ (see Definition 3.6). Hence the linear orders $\trianglelefteq'$ and $\trianglelefteq''$ on Δ_1 and Δ_2 are compatible. We currently lack an algorithmic way to check compatibility of linear orders. However, we have partial results that ease the search for compatibility in cases when the logic function M is additive (see Section 5).

Remark 4.7. Throughout Section 4 we will maintain the convention from Definition 4.6 that given two linear orders $\trianglelefteq'$ and $\trianglelefteq''$ on two sets U and V, respectively, then the partial order formed from their disjoint union over $U \cup V$ will be denoted $\trianglelefteq$.

Definition 4.8. Let $U := \Delta_0 \cup \ldots \cup \Delta_{m-1}$ and $V := \Delta_m$ for some $m \in \{1, \ldots, k\}$. *Block meshing* is the process of constructing the set of realizable linear orders on $U \cup V$ (see Algorithm 1). Note that when $m = k$, this amounts to constructing Υ.

Remark 4.9. For the remainder of this work, we will assume that any two elements $M(c_1, \ldots, c_k)$ and $M(d_1, \ldots, d_k)$ of $\mathcal{F}$ are distinct if there exists i such that $c_i = a_i$ and $d_i = b_i$, or vice versa. That is, elements with distinct a and b labelings are distinct, and so $|\mathcal{F}| = 2^k$. In other words, the partial order on $\mathcal{F}$ is strict ($<$), and orders on subsets of $\mathcal{F}$ are strict as well ($\triangleleft$). This assumption induces strictness on the block ordering as well:

$$\Delta_0 \prec \Delta_1 \ldots \prec \Delta_m.$$

To emphasize these orders' strictness, we will use notation without the equal bar throughout the rest of the work, as we have here.

The idea of block meshing naturally gives rise to the idea of positions within a linear order.

Definition 4.10. Let $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_m\}$ be subsets of $\mathcal{F}$ with $U \cap V = \emptyset$ and $U \prec V$. Let $\triangleleft'$ and $\triangleleft''$ be two linear orders on U and V, respectively. Assume without loss of generality that

$$u_1 \triangleleft' \cdots \triangleleft' u_n \text{ and } v_1 \triangleleft'' \cdots \triangleleft'' v_m.$$

Let $\triangleleft^*$ be a linear extension of the order $\triangleleft$ on $U \cup V$. For $i = 1, \ldots, n$, we say u_i is in the *position* p_j with respect to $\triangleleft^*$ if $v_j \triangleleft^* u_i \triangleleft^* v_{j+1}$. We say u_i is in position p_0 if $u_i \triangleleft^* v_1$, and u_i is in position p_m if $v_m \triangleleft^* u_i$.

Note it is possible for multiple elements of U to occupy the same position. For example, if $v_1 \triangleleft^* u_1 \triangleleft^* u_2 \triangleleft^* v_2$, we say u_1 and u_2 are in position p_1.

Definition 4.11. Let $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_m\}$ be subsets of $\mathcal{F}$, with $U \cap V = \emptyset$ and $U \prec V$. Let $\triangleleft'$ and $\triangleleft''$ be compatible linear orders on U and V, respectively. Then for each $u_i \in U$ let $\triangleleft'_i$ be the restriction of $\triangleleft'$ to $U \setminus \{u_i\}$ and let $\triangleleft''_i$ be a linear order on $V \cup \{u_i\}$ such that if $v_j \triangleleft'' v_k$, then $v_j \triangleleft''_i v_k$ as well. We define

$$\zeta_i := \left\{ j \;\middle|\; \text{if } u_i \text{ is in position } p_j, \text{ then } \triangleleft'_i \text{ and } \triangleleft''_i \text{ are compatible} \right\}.$$

We will sometimes say that u_i is in a *compatible position* p_j if $j \in \zeta_i$.

Algorithm 1 Construction of Υ

```
procedure Main(Δ_0, ..., Δ_k)
    Input: list of blocks
    Output: list of realizable linear extensions
    B ← Permutations(Δ_0)
    return BlockMeshing(0, B, ∅)
end procedure

procedure BlockMeshing(m, B, D)
    Global variables: Δ_0, ..., Δ_k
    Input: m ← block index
           B ← list of orders
           D ← accumulator
    Output: list of realizable linear extensions
    if m = k then
        D ← D ∪ B
        return D
    else
        for ◁′ ∈ B do
            for ◁″ ∈ Permutations(Δ_{m+1}) do
                B′ ← LinearOrders(◁′, ◁″, ∅)
                BlockMeshing(m + 1, B′, D)
            end for
        end for
    end if
    return D
end procedure

procedure LinearOrders(◁′, ◁″, B′)
    Input: ◁′ ← order, ◁″ ← order
           B′ ← accumulator
    Output: list of compatible orders
    U, V ← set of nodes in ◁′, ◁″
    if U = ∅ then
        return B′ ∪ {◁″}
    else
        for u ∈ U do
            ◁̃′ ← ◁′ restricted to U \ {u}
            for j = 0 to |V| do
                ◁̃″ ← insert u at index j of ◁″
                if Compatible(◁̃′, ◁̃″) then
                    B′ ← LinearOrders(◁̃′, ◁̃″, B′)
                end if
            end for
        end for
    end if
    return B′
end procedure
```

4.1 Construction Algorithm

We can now present the partial algorithm that we have devised to construct all of the linear extensions in Υ. As we mentioned previously, we are missing a key component of the algorithm, which is a procedure stating whether two orders are compatible. This is unfortunately a large hole, since it encompasses the algebraic constraints of the problem; however, we do have partial results for additive M, which we discuss in the following sections. Given that we lack the key component, the contribution of this algorithm is to leverage the unique block formation of the problem in order to construct the realizable linear extensions of the partial order.

In Algorithm 1, we introduce two recursive procedures BlockMeshing and LinearOrders. In BlockMeshing, we start with block Δ_0, which has exactly one element, $M(a_1, \ldots, a_k)$, and therefore exactly one linear order. Since the elements of Δ_1 are incomparable, we calculate the permutations of Δ_1 to check each one as a possible compatible order. Of course, every permutation of Δ_1 is compatible with Δ_0, as the (unique) compatible linear extension of $\Delta_0 \cup \Delta_1$ for each permutation simply has $M(a_1, \ldots, a_k)$ as the minimum element. However, this becomes a nontrivial question as soon as we consider Δ_m and Δ_{m+1} for $m > 0$.

Let B be the set of linear orders on $\bigcup_{i\leq m}\Delta_i$ such that for any j, any of the linear orders $\lhd'\in B$ restricted to the sets $\bigcup_{i\leq j}\Delta_i$ and $\bigcup_{j<i\leq k}\Delta_i$ are compatible. We submit each pair $(\lhd',\lhd'')$, with $\lhd'\in B$ and $\lhd''\in \mathsf{Permutations}(\Delta_{m+1})$, to the procedure $\mathsf{LinearOrders}$. Within this function, each element of $(\Delta_0\cup\ldots\cup\Delta_m,\lhd')$ is sequentially inserted into every position in Δ_{m+1} with respect to $\lhd''$, and each case is checked for compatibility. Each linear order on $\Delta_0\cup\ldots\cup\Delta_{m+1}$ formed from meshing $\lhd'$ into $\lhd''$, and that satisfies compatibility, is recorded.

$\mathsf{LinearOrders}$ is dependent on the procedure $\mathsf{Compatible}$, for which we do not possess an explicit algorithm. $\mathsf{Compatible}(\lhd',\lhd'')$ returns True if $\lhd'$ and $\lhd''$ are compatible and False otherwise. Provided that such a procedure can be defined, perhaps via a symbolic mathematics language, we prove the following theorem.

Theorem 4.12. *Assuming a correct algorithm for* $\mathsf{Compatible}$ *and a strictly ordered linear extension (that is, no equality between distinct elements as in Remark 4.9), Algorithm 1 records each realizable linear extension of the partial order $<$ on $\mathcal{F}$ exactly once.*

Proof. Let $<^*$ be a realizable linear extension of $<$. Then for $m=0,\ldots,k-1$, letting $U=\Delta_0\cup\ldots\cup\Delta_m$ and $V=\Delta_{m+1}$, $<_U := <^*\big|_U$ and $<_V := <^*\big|_V$ are by definition compatible linear orders on U and V. Now we proceed as follows.

Let $m=0$. Since $U=\Delta_0=\{M(a_1,\ldots,a_k)\}$, $B=\mathsf{Permutations}(\Delta_0)$ contains the single permutation of U, which is trivially $<^*\big|_U$. We have $m\neq k$, so for the single element $<^*\big|_U$ of B, we compute $\mathsf{Permutations}(\Delta_1)$. By definition, $<^*\big|_V\in\mathsf{Permutations}(\Delta_1)$, so we compute $L=\mathsf{Compatible}\left(<^*\big|_U,<^*\big|_V\right)$, and as previously noted, $L=\mathsf{True}$. Therefore we define $B'=\mathsf{LinearOrders}\left(<^*\big|_U,<^*\big|_V,\emptyset\right)$. Since $\mathsf{LinearOrders}$ checks every $u\in U$ at every place in $\lhd''$, we will recover $<^*\big|_{U\cup V}$ in the output accumulator B' of $\mathsf{LinearOrders}$.

Now suppose for some $m\in\{0,\ldots,k-1\}$, for each $n\leq m$, $\mathsf{BlockMeshing}(n,B,D)$ computes $<^*\big|_{U'}$ for $U'=\Delta_0\cup\ldots\cup\Delta_n$. Let $U=\Delta_0\cup\ldots\cup\Delta_m$ and $V=\Delta_{m+1}$. Since we have $<^*\big|_U\in B$ by assumption and $<^*\big|_V\in\mathsf{Permutations}(\Delta_{m+1})$, and $L=\mathsf{Compatible}\left(<^*\big|_U,<^*\big|_V\right)=\mathsf{True}$, we then compute $B'=\mathsf{LinearOrders}\left(<^*\big|_U,<^*\big|_V,\emptyset\right)$, and, for the same reason as above, $<^*\big|_{U\cup V}\in B'$. Hence when $m=k-1$, $\mathsf{BlockMeshing}$ will compute $<^*$. Moreover, since $\mathsf{BlockMeshing}$ takes a set union at every complete order, the linear extension $<^*$ is recorded exactly once. Since we assume a strict order (see Remark 4.9), none of the distinct recorded sequences are equivalent. $\square$

Remark 4.13. It remains an open problem to determine the oracle-time complexity of Algorithm 1 taking the unsolved decision question $\mathsf{Compatible}$ to be an oracle.

4.2 Counting Theorems

Given compatible linear orders $\lhd'$ and $\lhd''$ on two disjoint sets, each equal to some union of blocks, we will now show how to compute an upper bound on the number of realizable linear extensions of $\lhd$.

Let $U=\{u_1,\ldots,u_n\}$ and $V=\{v_1,\ldots,v_m\}$ be subsets of $\mathcal{F}$ with $U\cap V=\emptyset$ and $U\prec V$. Consider two compatible orders $\lhd'$ and $\lhd''$ on U and V, respectively, such that

$$u_1\lhd'\cdots\lhd' u_n \text{ and } v_1\lhd''\cdots\lhd'' v_m,$$

and let $\lhd$ be the partial order on $U\cup V$ equal to the disjoint union of $\lhd'$ and $\lhd''$ as in Definition 4.6. Recall ζ_i is given in Definition 4.11. Let $j_0=0$ and recursively define for $i=1,\ldots,n$

$$z_i=\zeta_i\cap\{l\mid l\geq j_{i-1}\},\tag{3}$$

where $j_{i-1} \in z_{i-1}$ is chosen at each step before computing z_i. Notice that $z_1 = \zeta_1$. After the construction all the way through z_n, notice that each u_i is fixed in a position p_{j_i} that independently guarantees the existence of a linear extension consistent with the standard order on $\mathbb{R}$. Moreover, the choices $\{j_i\}$ jointly satisfy $u_1 \lhd' \cdots \lhd' u_n$. This is not necessarily enough to state that there is a linear extension jointly satisfying the collective choice of $\{j_i\}$, but it does provide an upper bound on the number of realizable linear extensions of $\lhd$.

Lemma 4.14. *For $i = 1, \ldots, n-2$, recursively fix $j_i \in z_i$. Then the number of realizable linear extensions of $\lhd$ that can be constructed by varying positions $p_{j_{n-1}}$ and p_{j_n} for u_{n-1} and u_n is at most*

$$\sum_{j_{n-1} \in z_{n-1}} |z_n|.$$

Proof. Let $p_{j_1} \leq \cdots \leq p_{j_{n-2}}$ be fixed compatible positions for $u_1, \ldots, u_{n-2}$, respectively, and let $p_{j_{n-1}}$ and p_{j_n} vary. By definition, u_{n-1} is in a compatible position $p_{j_{n-1}}$ only if $p_{j_{n-1}} \in \zeta_{n-1}$. Moreover, $u_{n-2} \lhd' u_{n-1}$ implies that $j_{n-2} \leq j_{n-1}$ under any extension of $\lhd$. Therefore, $j_{n-1} \in z_{n-1}$ is necessary to construct a realizable linear extension. Given that u_{n-1} is in some position $p_{j_{n-1}}$, we then know that $j_n \in z_n$ by a similar argument. Then the number of realizable linear extensions is at most the sum of the size of sets z_n over all $j_{n-1} \in z_{n-1}$ as desired. ☐

Theorem 4.15. *Let $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_m\}$ be subsets of $\mathcal{F}$ with $U \cap V = \emptyset$ and $U \prec V$. Consider two compatible orders $\lhd'$ and $\lhd''$ on U and V, respectively, such that*

$$u_1 \lhd' \cdots \lhd' u_n \text{ and } v_1 \lhd'' \cdots \lhd'' v_m.$$

Then the number of realizable linear extensions of the order $\lhd$ on $U \cup V$ is at most

$$\sum_{j_1 \in z_1} \sum_{j_2 \in z_2} \cdots \sum_{j_{n-1} \in z_{n-1}} |z_n|. \tag{4}$$

Proof. For a proof by induction, we first let $n = 2$. By Lemma 4.14, the number of pairs of positions that u_1 and u_2 can occupy is at most

$$\sum_{j_1 \in z_1} |z_2|.$$

For the inductive step, assume we have $U = \{u_1, u_2, \ldots, u_{n-1}, u_n\}$, and that for each subset $U' = \{u_1, u_2, \ldots, u_m\}$, with $m \leq n-1$, it is true that the number of realizable linear extensions of $\lhd \big|_{U' \cup V}$ is at most

$$\sum_{j_1 \in z_1} \sum_{j_2 \in z_2} \cdots \sum_{j_{m-1} \in z_{m-1}} |z_m| \quad = \quad \sum_{j_1 \in z_1} \sum_{j_2 \in z_2} \cdots \sum_{j_{m-1} \in z_{m-1}} \sum_{j_m \in z_m} 1. \tag{5}$$

This is in particular true for $m = n-2$. Then Lemma 4.14 states that the number of realizable linear extensions of $\lhd$ with fixed positions j_i for $i = 1, \ldots, n-2$ is at most

$$\sum_{j_{n-1} \in z_{n-1}} |z_n|. \tag{6}$$

The total number of distinct collections $\{j_i\}_{i=1}^{n-2}$ for which this count can be made is (5) with $m = n-2$. Therefore, the total is at most the sum of counts over all of these choices, which we obtain by replacing 1 in the right hand side of (5) (with $m = n-2$) with (6):

$$\sum_{j_1 \in z_1} \sum_{j_2 \in z_2} \cdots \sum_{\substack{j_{n-2} \in \\ z_{n-2}}} \sum_{\substack{j_{n-1} \in \\ z_{n-1}}} |z_n|.$$

☐

5 Additive logic M

There is a special case of (1) that is of particular interest, namely the case when the logic function M is additive, so that $M(c_1, \dots, c_k) = \sum_{i=1}^{k} c_i$. Throughout Section 5, we will assume M is additive, unless otherwise stated, and we will prove stronger results than in the previous section for general multilinear M.

Remark 5.1. For a fixed k, the purely additive and purely multiplicative cases, e.g. $\sigma_1 + \sigma_2 + \sigma_3$ and $\sigma_1\sigma_2\sigma_3$, are effectively the same. Let $\mathcal{F}_+$ and $\mathcal{F}_\times$ refer to $\mathcal{F}$ in the additive and multiplicative cases, respectively. Then for any $\leq_+ \in \Upsilon$ in the additive case, the natural logarithm provides an order isomorphism from $(\mathcal{F}_+, \leq_+)$ to a unique $(\mathcal{F}_\times, \leq_\times)$ in the multiplicative case. Similarly, the exponential function provides an order isomorphism from each $(\mathcal{F}_\times, \leq_\times)$ to a unique $(\mathcal{F}_+, \leq_+)$.

In the additive case additional symmetries simplify the process of finding compatible linear orders. We make the following conjecture.

Conjecture 5.2. Fix a particular $k \in \mathbb{N}$ and let Υ_+ be the set of realizable linear extensions of the partial order $<$ with additive M and Υ_a that for an arbitrary logic function, both with $|S(j)| = k$. Then

$$|\Upsilon_+| \leq |\Upsilon_a|.$$

Our reasoning for this conjecture is as follows. Recall the poset in Figure 2a. Note that in this example and all examples where M is additive, each edge in the Hasse diagram is characterized by a difference $b_i - a_i$ for some $i \in \{1, \dots, k\}$, with no multiplicative dependence on other terms. This imposes strong requirements for compatibility of linear orders, as we will show in this section.

Definition 5.3. Let $A, B \subset \{1, \dots, k\}$, $A \cup B = \{1, \dots, k\}$, and $A \cap B = \emptyset$. Let $c = \sum_{i\in A} a_i + \sum_{i \in B} b_i \in \mathcal{F}$. Then the *complement* of c is

$$c' = \sum_{i \in A} b_i + \sum_{i \in B} a_i.$$

It is clear from Definition 5.3 that $c \in \Delta_\ell$ if and only if $c' \in \Delta_{k-\ell}$. Furthermore, $c'' = c$. This leads to the following result.

Theorem 5.4. *Let $c \in \mathcal{F}$, $r \in \{1, 2, 3, \dots, 2^k\}$, and let $<^*$ be a realizable linear extension of $<$ such that c is in the r^{th} place of $(\mathcal{F}, <^*)$. Then the complement of c is in the $(2^k + 1 - r)^{th}$ place of $(\mathcal{F}, <^*)$.*

Proof. Because $<^*$ is realizable, it must respect the arithmetic of $\mathbb{R}$. Let $c_a := a_1 + \cdots + a_k$ and $c_b := b_1 + \cdots + b_k$. Then for some $B \subset \{1, \dots, k\}$ we have $c = c_a + \sum_{i \in B}(b_i - a_i)$, and $c' = c_b - \sum_{i \in B}(b_i - a_i)$. This implies $c' = c_b - (c - c_a)$, and therefore $c' + c = c_b + c_a$. Because the right hand side of this last equation is independent of r, we know that for each $d \in \mathcal{F}$, $d <^* c$ if and only if $c' <^* d'$. This implies that since exactly $r - 1$ elements $d \in \mathcal{F}$ satisfy $d <^* c$, there must be exactly $r - 1$ elements $d' \in \mathcal{F}$ satisfying $c' <^* d'$. $\square$

We have the following immediate corollary of this result.

Theorem 5.5. *For each $\ell \in \{1, \dots, k\}$,*

1. *Each linear order on Δ_ℓ is equivalent to a linear order on $\Delta_{k-\ell}$.*
2. *Each linear order on $\Delta_0 \cup \ldots \cup \Delta_{\lfloor \frac{k}{2} \rfloor}$ implies a unique linear order on $\Delta_{\lfloor \frac{k}{2} \rfloor + 1} \cup \ldots \cup \Delta_k$.*

We seek compatible orders on sets

$$U = \Delta_0 \cup \ldots \cup \Delta_{m-1} \quad \text{and} \quad V = \Delta_m$$

for additive M. We now show that two compatible positions for an element $u_i \in U$ interpolate to an intermediate position.

Lemma 5.6. *Let $(U, \lhd')$ and $(V, \lhd'')$ be two linearly ordered subsets of $\mathcal{F}$ with $U \cap V = \emptyset$ and $U \prec V$, and consider the poset $(U \cup V, \lhd)$. Let $u_i \in U$ and suppose there are two realizable linear extensions $\lhd_0^*$ and $\lhd_2^*$ of $\lhd$ such that u_i occupies positions p_j and p_{j+r} for some $r \geq 2$ with respect to $\lhd_0^*$ and $\lhd_2^*$, respectively. Then for any $1 \leq l \leq r-1$ there exists another realizable linear extension $\lhd_1^*$ of $\lhd$ under which u_i occupies position p_{j+l}.*

Proof. Let $n = |U|$ and $m = |V|$ and denote the elements of U and V by $u_1 \lhd \cdots \lhd u_n$ and $v_1 \lhd \cdots \lhd v_m$. Note this implies all of

$$u_1 \lhd_0^* \cdots \lhd_0^* u_n; \quad u_1 \lhd_2^* \cdots \lhd_2^* u_n; \quad v_1 \lhd_0^* \cdots \lhd_0^* v_m; \quad v_1 \lhd_2^* \cdots \lhd_2^* v_m.$$

By assumption there exist tuples $p, q \in \mathbb{R}^{2k}$ at which $\lhd_0^*$ and $\lhd_2^*$ are realized, respectively. We will denote each $u \in U$ and $v \in V$ by $u(p)$ and $v(p)$ or $u(q)$ and $v(q)$ to indicate the parameter at which they are realized. Then the hypotheses imply that

$$u_i(p) \lhd_0^* v_{j+1}(p) \quad \text{and} \quad v_{j+r}(q) \lhd_2^* u_i(q).$$

Because $\lhd_0^*$ and $\lhd_2^*$ are realizable, they must respect the standard linear order on $\mathbb{R}$; for the remainder of this proof we will abuse notation and denote this order by $\leq$. We leverage this fact in the remainder of the proof, starting with

$$u_i(p) < v_{j+1}(p) \quad \text{and} \quad v_{j+r}(q) < u_i(q). \tag{7}$$

We define the line segment $t : [0,1] \to \mathbb{R}^{2k}$ from q to p in parameter space such that for all $s \in [0,1]$, $t(s) = sp + (1-s)q$. In particular, note that if $p = (a_1(p), b_1(p), \ldots, a_k(p), b_k(p))$ and $q = (a_1(q), b_1(q), \ldots, a_k(q), b_k(q))$, then

$$t(s)_\ell = \begin{cases} sa_c(p) + (1-s)a_c(q) & \text{if } \ell = 2c-1 \\ sb_c(p) + (1-s)b_c(q) & \text{if } \ell = 2c \end{cases} \tag{8}$$

We may then note that $0 < t(s)_\ell < t(s)_{\ell+1}$ for $\ell = 1, \ldots, 2k-1$, so that the quantity $t(s)$ satisfies the requirements of a parameter for any $s \in [0,1]$; see Definiton 3.6. In other words,

$$\begin{aligned} t(s) &= (a_1(t(s)), b_1(t(s)), \ldots, a_k(t(s)), b_k(t(s))) \\ &\text{and } 0 < \min\{a_c(p), a_c(q)\} \leq a_c(t(s)) < b_c(t(s)) \quad \text{for all } c \in \{1, \ldots, k\}. \end{aligned}$$

Note that by the additivity of M, we have

$$\begin{aligned} &M(sc_1(p) + (1-s)c_1(q), \ldots, sc_k(p) + (1-s)c_k(q)) \\ &\qquad = sM(c_1(p), \ldots, c_k(p)) + (1-s)M(c_1(q), \ldots, c_k(q)). \end{aligned} \tag{9}$$

Then as a consequence of (8) and (9), for $\alpha = 1, \ldots, n$ and $\beta = 1, \ldots, m$ and every $s \in [0,1]$, we have

$$\begin{aligned} u_\alpha(t(s)) &= su_\alpha(p) + (1-s)u_\alpha(q) \\ v_\beta(t(s)) &= sv_\beta(p) + (1-s)v_\beta(q). \end{aligned} \tag{10}$$

Since for $\alpha = 1, \ldots, n$ and $\beta = 1, \ldots, m$ and every $s \in [0,1]$, $u_\alpha(t(s))$ and $v_\beta(t(s))$ are fixed real numbers, the set $U \cup V$ evaluated at $t(s)$ is endowed with the restriction of the standard order $\leq$ on $\mathbb{R}$. Generically, the restriction is strict. If the restriction is not strict, an arbitrarily small perturbation of s will result in a strict order, since there are only finitely many $s \in [0,1]$ at which $w_\eta(t(s)) = w_\gamma(t(s))$ occurs, where $w_\eta, w_\gamma \in U \cup V$. Only finitely many such s exist because, by (10), each $w(t(s))$ is a strictly monotone function of s and we impose $w_\eta \neq w_\gamma$ at both p, q by Remark 4.9. This means $w_\eta(t(s))$ and $w_\gamma(t(s))$ can only intersect at a single point, not along an interval.

Moreover, if $w_\eta \lhd w_\gamma$, then $w_\eta(p) < w_\gamma(p)$ and $w_\eta(q) < w_\gamma(q)$, so it is immediate from (10) that

$$w_\eta(t(s)) < w_\gamma(t(s))$$

for any $s \in [0,1]$. Thus the strict order $\lhd$ on $U \cup V$ is preserved under $t(s)$, and the (generically strict) restriction of $\leq$ on $\mathbb{R}$ to $U \cup V$ evaluated at $t(s)$ is a realizable linear extension of $\lhd$.

It remains to show that for each position p_{j+l} for $l \in \{1, \ldots, r-1\}$, there exists a realizable linear extension $\lhd_1^*$ induced by $t(s_0)$ for some $s_0 \in [0,1]$ such that $u_i(t(s_0))$ is in position p_{j+l}. We do this by choosing an arbitrary $l \in \{1, \ldots, r-1\}$ and applying a bisection technique.

Define $F : [0,1] \to \mathbb{R}$ such that for all $s \in [0,1]$,

$$F(s) = 2u_i(t(s)) - v_{j+r}(t(s)) - v_{j+1}(t(s)).$$

Then F is a continuous function such that

$$\begin{aligned} F(0) &= 2u_i(p) - v_{j+1}(p) - v_{j+r}(p) = (u_i(p) - v_{j+1}(p)) + (u_i(p) - v_{j+r}(p)) < 0 \\ F(1) &= 2u_i(q) - v_{j+1}(q) - v_{j+r}(q) = (u_i(q) - v_{j+1}(q)) + (u_i(q) - v_{j+r}(q)) > 0, \end{aligned}$$

which follows from (7). Therefore there exists $s_0 \in (0,1)$ such that $F(s_0) = 0$. This means

$$\frac{1}{2}\left(v_{j+1}(t(s_0)) + v_{j+r}(t(s_0))\right) = u_i(t(s_0)).$$

Since the average of two unequal numbers lies strictly between them, we have

$$v_{j+1}(t(s_0)) < u_i(t(s_0)) < v_{j+r}(t(s_0)). \tag{11}$$

Generically, $u_i(t(s_0)) \neq v_{j+w}(t(s_0))$ for any $w \in \{1, \ldots, r-1\}$, and even in the degenerate cases we can use continuity to perturb s_0 by an arbitrarily small amount such that $u_i(t(s_0)) \neq v_{j+w}(t(s_0))$ for any $w \in \{1, \ldots, r-1\}$. Therefore, by (11), under the linear extension of $\lhd$ realized at $t(s_0)$, u_i is in some position p_{j+w} where $w \in \{1, 2, \ldots, r-1\}$.

Now if $l = w$ the result holds. If $l < w$, then we repeat the argument with r replaced by w, and if $w < l$, then we repeat the argument with j replaced by $j+w$. We continue to repeat the argument in this way; because V is a finite set, there are only finitely many positions between p_j and p_{j+r}, and each time we repeat the argument we show that there exists a realizable linear extension of $\lhd$ under which u_i occupies some new such position. Hence this process must eventually produce a realizable linear extension $\lhd_1^*$ of $\lhd$ under which u_i occupies p_{j+l}. $\square$

Remark 5.7. Theorem 5.5 and Lemma 5.6 are two properties of additive logic that substantially ease the search for compatible orders. Consequently, it would be beneficial to show these properties hold for general logic functions M. It remains open whether Lemma 5.6 holds for non-additive M, and because the proof of the lemma depends on linearity of M, it does not immediately apply to any M involving multiplication. On the other hand, Theorem 5.5 may fail when M is not additive.

As a counterexample, let $k = 3$ and let $M(x_1, x_2, x_3) = (x_1 + x_2)x_3$. Define $a_1 = 1$, $a_2 = 4$, $a_3 = 9$, $b_1 = 7$, $b_2 = 8$, and $b_3 = 10$. It is readily seen that

$$(a_1 + a_2)b_3 < (a_1 + b_2)a_3 < (b_1 + a_2)a_3$$

and

$$(a_1 + b_2)b_3 < (b_1 + a_2)b_3 < (b_1 + b_2)a_3$$

Now let $a_1 = 4$, $a_2 = 8$, $a_3 = 12$, $b_1 = 14$, $b_2 = 16$, and $b_3 = 17$. Then we again have

$$(a_1 + a_2)b_3 < (a_1 + b_2)a_3 < (b_1 + a_2)a_3$$

yet now

$$(a_1 + b_2)b_3 < (b_1 + b_2)a_3 < (b_1 + a_2)b_3.$$

Hence the order $(a_1 + a_2)b_3 < (a_1 + b_2)a_3 < (b_1 + a_2)a_3$ on Δ_1 does not imply any single order on Δ_2.

We now discuss finding sets ζ_i, as in Definition 4.11, for additive M. Let $U := \Delta_0 \cup \ldots \cup \Delta_{m-1}$ and $V := \Delta_m$, and let $\lhd'$ and $\lhd''$ be compatible linear orders on U and V, respectively, and let $\lhd$ be the partial order on $U \cup V$ formed by the disjoint union of $\lhd'$ and $\lhd''$. Let $u_i \in U$. We wish to constrain the set of positions that u_i can occupy by containing the set ζ_i. Let $c_a := a_1 + \cdots + a_k$. From Definition 5.3 there exist disjoint (possibly empty) sets $A, B \subset \{1, \ldots, k\}$ such that $u_i = \sum_{j \in A} a_j + \sum_{j \in B} b_j \in \mathcal{F}$, or equivalently, $u_i = c_a + \sum_{j \in B}(b_j - a_j)$. We decompose the index set B into two disjoint (possibly empty) sets B_1 and B_2 such that $B = B_1 \cup B_2$. Let $\left\{1, 2, 3, \ldots, 2^{|B|}\right\}$ be an indexing set for all such decompositions, and for each $\alpha \in \left\{1, 2, 3, \ldots, 2^{|B|}\right\}$, define

$$w_1^\alpha := \sum_{j \in B_1^\alpha} (b_j - a_j) \text{ and } w_2^\alpha := \sum_{j \in B_2^\alpha} (b_j - a_j).$$

Then for each $\alpha \in \left\{1, \ldots, 2^{|B|}\right\}$ we have a representation of u_i as

$$u_i = c_a + w_1^\alpha + w_2^\alpha.$$

Since $u_i \in U$, we have $|B_1^\alpha|, |B_2^\alpha| \leq m-1$, which implies that both $c_a + w_1^\alpha$ and $c_a + w_2^\alpha$ are elements of U.

Definition 5.8. As above, let $U := \Delta_0 \cup \ldots \cup \Delta_{m-1}$ and $V := \Delta_m$, and let $\lhd'$ and $\lhd''$ be compatible linear orders on U and V, respectively, and let $\lhd$ be the partial order on $U \cup V$ formed by the disjoint union of $\lhd'$ and $\lhd''$. Let $u_i = c_a + \sum_{j \in B}(b_j - a_j) = c_a + w_1^\alpha + w_2^\alpha \in U$ where B_1^α, B_2^α, $\alpha \in \left\{1, 2, 3, \ldots, 2^{|B|}\right\}$, is a decomposition of B as above. Let

$$\mathcal{L}^\alpha := \{\lambda \mid c_a + \lambda \in U \setminus \{c_a + w_1^\alpha\} \text{ and } c_a + \lambda + w_2^\alpha \in \Delta_m\}.$$

Since for all $\lambda \in \mathcal{L}^\alpha$, $c_a + \lambda$ and $c_a + w_1^\alpha$ are distinct elements of U, and because $\lhd'$ is a strict linear order on U, we must have either $c_a + w_1^\alpha \lhd' c_a + \lambda$ or $c_a + \lambda \lhd' c_a + w_1^\alpha$. For each $\lambda \in \mathcal{L}^\alpha$, define a partial order $\lhd_\lambda^\alpha$ on $\Delta_m \cup \{u_i\}$ such that $u_i \lhd_\lambda^\alpha c_a + \lambda + w_2^\alpha$ if and only if $c_a + w_1^\alpha \lhd' c_a + \lambda$, and $c_a + \lambda + w_2^\alpha \lhd_\lambda^\alpha u_i$ if and only if $c_a + \lambda \lhd' c_a + w_1^\alpha$, and all other elements of $\Delta_m \cup \{u_i\}$ are unordered by $\lhd_\lambda^\alpha$. Now define $\lhd^\alpha$ to be the partial order on $\Delta_m \cup \{u_i\}$ equal to the disjoint union $\bigcup_{\lambda \in \mathcal{L}^\alpha} \lhd_\lambda^\alpha$.

Lemma 5.9. *If $\lhd^*$ is a realizable linear extension of the order $\lhd$ on $U \cup V$, then for all $\alpha \in \left\{1, 2, 3, \ldots, 2^{|B|}\right\}$ and $\lambda \in \mathcal{L}^\alpha$, $u_i \lhd^* c_a+\lambda+w_2^\alpha$ if and only if $u_i \lhd^\alpha c_a+\lambda+w_2^\alpha$, and $c_a+\lambda+w_2^\alpha \lhd^* u_i$ if and only if $c_a + \lambda + w_2^\alpha \lhd^\alpha u_i$.*

Proof. Let $\alpha \in \left\{1, 2, 3, \ldots, 2^{|B|}\right\}$. Since $\lhd$ has a realizable extension $\lhd^*$, $\lhd$ and $\lhd^*$ must respect the strict linear order on $\mathbb{R}$, which for this proof we will again denote by $<$.

First let $u_i \lhd^* c_a + \lambda + w_2^\alpha$ where $\lambda \in \mathcal{L}^\alpha$. Then $u_i < c_a + \lambda + w_2^\alpha$. Then by the translational property of $<$, $c_a + w_1^\alpha < c_a + \lambda$. Then $c_a + w_1^\alpha \lhd' c_a + \lambda$, since $\lhd$ must respect $<$ and $\lhd' = \lhd\big|_U$ and $c_a + w_1^\alpha, c_a + \lambda \in U$. This implies $u_i \lhd_\lambda^\alpha c_a + \lambda + w_2^\alpha$ by definition of $\lhd_\lambda^\alpha$. Since $\lhd^\alpha$ is the disjoint union of all $\lhd_\lambda^\alpha$, this implies $u_i \lhd^\alpha c_a + \lambda + w_2^\alpha$. An analogous argument shows that if $c_a + \lambda + w_2^\alpha \lhd^* u_i$, then $c_a + \lambda + w_2^\alpha \lhd^\alpha u_i$.

Now let $u_i \lhd^\alpha c_a + \lambda + w_2^\alpha$ where $\lambda \in \mathcal{L}^\alpha$, so that $u_i \lhd_\lambda^\alpha c_a + \lambda + w_2^\alpha$ by definition of $\lhd^\alpha$. Then $c_a + w_1^\alpha \lhd' c_a + \lambda$ by definition of $\lhd_\lambda^\alpha$, and $c_a + w_1^\alpha < c_a + \lambda$ by realizability as before, implying $u_i < c_a + \lambda + w_2^\alpha$ by translation. Finally $u_i \lhd^* c_a + \lambda + w_2^\alpha$ since $\lhd^* = <|_{U \cup V}$ by realizability. An analogous argument shows that if $c_a + \lambda + w_2^\alpha \lhd^\alpha u_i$, then $c_a + \lambda + w_2^\alpha \lhd^* u_i$. $\square$

Definition 5.10. Let $\alpha \in \left\{1, 2, 3, \ldots, 2^{|B|}\right\}$. A *shifted order* $U_\alpha = U + w_2^\alpha$ is the set of all $u + w_2^\alpha$ where $u \in U$. In other words, U_α is the set of all $c_a + \lambda + w_2^\alpha$ where $c_a + \lambda \in U$. If they exist, let

$$m_\alpha := \max_{\lhd''}\{v_j \in \Delta_m \cap U_\alpha \mid v_j \lhd^\alpha u_i\}$$
$$M_\alpha := \min_{\lhd''}\{v_j \in \Delta_m \cap U_\alpha \mid u_i \lhd^\alpha v_j\}$$

where the maximum and minimum are with respect to $\lhd''$. When these exist, we define $p_{\min^\alpha}$ and $p_{\max^\alpha}$ to be the positions in Δ_m immediately above m_α and immediately below M_α (with respect to $\lhd''$), respectively. Now we define

$$\mathcal{P}_i^\alpha := \{j \mid \min^\alpha \leq j \leq \max^\alpha\}.$$

If either m_α or M_α does not exist (which may occur if, for instance, $U_\alpha \cap \Delta_m = \emptyset$), we remove the respective inequality from the definition of $\mathcal{P}_i^\alpha$. If neither exists, then $\mathcal{P}_i^\alpha$ is just the index set of all positions among the elements of Δ_m. By construction, $\mathcal{P}_i^\alpha$ is the index set of the positions that u_i may occupy under the order $\lhd^\alpha$ on $\Delta_m \cup \{u_i\}$.

We now have the following immediate result.

Corollary 5.11. *Let $\mathcal{P}_i := \bigcap_\alpha \mathcal{P}_i^\alpha$. Then $\zeta_i \subset \mathcal{P}_i$, where recall from Definition 4.11, ζ_i is defined with respect to $\lhd$. In other words, under any realizable linear extension of $\lhd$, u_i cannot occupy any position whose index is not in $\mathcal{P}_i$.*

Proof. By assumption $\lhd'$ and $\lhd''$ are compatible, so there exists at least one realizable linear extension $\lhd^*$ of the order $\lhd$ on $U \cup V$. By Lemma 5.9, $\lhd^\alpha$ must respect $\lhd^*$ for all $\alpha \in \left\{1, 2, 3, \ldots, 2^{|B|}\right\}$. Then since under $\lhd^\alpha$, the index of the position of u_i must be in $\mathcal{P}_i^\alpha$, this must hold under $\lhd^*$ as well. $\square$

Conjecture 5.12. If M is additive and $|\mathcal{F}| = 2^k$ as assumed in Remark 4.9, then $\zeta_i = \mathcal{P}_i$.

The proof of the conjecture requires that restrictions that an order $\lhd'$ on U imposes on any compatible order $\lhd''$ on V can be captured by a shift by w_2^α for some α. Although we do not have a proof of this, the partial result in Corollary 5.11 restricts our search for compatible positions to a smaller set.

6 Additive examples

In this and the following section we compute an upper bound on $|\Upsilon|$ for a few specific functions f. Roughly speaking, we do this by applying Algorithm 1; however, because we lack a conclusive method to check compatibility short of finding explicit parameters, we will not compute the exact sets ζ_i of compatible positions. Instead, we will compute sets η_i containing these ζ_i. For each pair $U = \Delta_0 \cup \ldots \cup \Delta_m$ and $V = \Delta_{m+1}$, we construct each η_i by obtaining upper and lower bounds on the compatible positions of each element of U; these bounds are determined by the algebraic constraints from the logic M and certain considerations of compatibility. However, the indices of the maximum and minimum positions in each η_i may not actually be in the corresponding ζ_i, and therefore even in an additive system the result of Lemma 5.6 may not hold for our calculated η_i. Because of this, the upper bounds we compute for these examples may be less strict than the respective upper bounds given by Theorem 4.15. However, in many cases in the following examples, we have observed that $\eta_i = \zeta_i = \{0\}$.

For any case with $k \leq 2$, counting realizable linear extensions of the partial order $<$ is trivial, as these cases do not require any consideration of compatibility. In particular, when $k = 1$, Υ contains only the linear extension $<^*$ such that $a_1 <^* b_1$, and when $k = 2$, Υ contains two linear extensions, $<_1^*$ and $<_2^*$, such that

$$a_1 + a_2 <_1^* b_1 + a_2 <_1^* a_1 + b_2 <_1^* b_1 + b_2$$
$$a_1 + a_2 <_2^* a_1 + b_2 <_2^* b_1 + a_2 <_2^* b_1 + b_2$$

These two orders are realized at $(a_1, b_1, a_2, b_2) = (1, 2, 1, 3)$ and $(1, 3, 1, 2)$, respectively. Recall from Remark 5.1 that $M(c_1, c_2) = c_1 + c_2$ has an order isomorphism to $M(c_1, c_2) = c_1c_2$, so that the multiplicative case has two linear extensions as well.

In addition to these cases, we can bound $|\Upsilon|$ for f with additive M and $k = 3$ or 4 and for f with non-additive M and $k = 3$. Although $|\Upsilon|$ has the trivial upper bound of $|W|$ (the number of all linear extensions of $<$), our bound is more strict. In particular, for the $k = 3$ examples below, $|W| = 48$ in comparison to $|\Upsilon| \leq 12$ (Section 6.1) and $|\Upsilon| \leq 20$ (Section 7). More dramatically, in the additive case where $k = 4$, $|W| = 1{,}680{,}384$ as opposed to $|\Upsilon| \leq 336$ (Section 6.2). The numbers $|W|$ for $k = 3$ and $k = 4$ were calculated using the digraphtools package for Python 2.7.

Remark 6.1. In the following examples, we largely neglect the minimal and maximal elements $M(a_1, \ldots, a_k)$ and $M(b_1, \ldots, b_k)$ of $\mathcal{F}$, since their exclusion does not affect the counts we make.

6.1 Additive M with $k = 3$

In this section, refer back to Figures 2a and 3. Consider the blocks

$$\begin{aligned}
\Delta_0 &= \{a_1 + a_2 + a_3\}\\
\Delta_1 &= \{b_1 + a_2 + a_3, a_1 + b_2 + a_3, a_1 + a_2 + b_3\}\\
\Delta_2 &= \{b_1 + b_2 + a_3, b_1 + a_2 + b_3, a_1 + b_2 + b_3\}\\
\Delta_3 &= \{b_1 + b_2 + b_3\}.
\end{aligned}$$

We will begin by choosing a linear order $\lhd'$ on Δ_1. We will arbitrarily pick $\lhd'$ such that

$$b_1 + a_2 + a_3 \lhd' a_1 + b_2 + a_3 \lhd' a_1 + a_2 + b_3.$$

By Theorem 5.5, this implies a unique compatible linear order $\lhd''$ on Δ_2, such that

$$b_1 + b_2 + a_3 \lhd'' b_1 + a_2 + b_3 \lhd'' a_1 + b_2 + b_3.$$

Now we must find sets of positions containing ζ_i for each $u_i \in \Delta_1$. Because $|\Delta_1| = 3$, we require sets for each of ζ_1, ζ_2, and ζ_3. Note the partial order $<$ on $\mathcal{F}$ stipulates

$$b_1 + a_2 + a_3 < a_1 + b_2 + a_3 < b_1 + b_2 + a_3$$

and

$$a_1 + a_2 + b_3 < b_1 + a_2 + b_3,$$

but $a_1 + a_2 + b_3$ and $b_1 + b_2 + a_3$ are not comparable under $<$. Therefore, $\zeta_1 = \eta_1 := \{0\}$, $\zeta_2 = \eta_2 := \{0\}$ and $\zeta_3 \subset \eta_3 := \{0, 1\}$.

Finally, we must extend $\lhd'$ and $\lhd''$ to $\Delta_1 \cup \Delta_2$. We will substitute our sets containing η_1, η_2, η_3 into (4); for $i = 2, 3$, let $h_i := \eta_i \cap \{l \mid l \geq j_{i-1}\}$ where recall j_{i-1} refers to the position of u_{i-1} (note the analogy with (3)). Then

$$\sum_{j_1 \in \{0\}} \sum_{j_2 \in h_2} |h_3| = 2$$

Therefore, there exist at most two realizable linear extensions of $(\mathcal{F}, <)$ that respect our linear order $\lhd'$ on Δ_1. Since this assumption was an arbitrary choice of one of 3! permutations of the elements on Δ_1, there exist at most $3! \cdot 2 = 12$ realizable linear extensions of $<$. Each of these can be found by renaming parameters in the linear extensions we have computed above.

6.2 Additive M with $k = 4$

The poset $(\mathcal{F}, <)$ is illustrated in Figure 4. Because linear orders on these blocks and their unions are lengthy to write, we will use the shorthand in Figure 5 to refer to the vertices in Figure 4.

We divide $\mathcal{F}$ into blocks as follows.

$$\begin{aligned}
\Delta_0 &= \{v_0\} \\
\Delta_1 &= \{v_1, v_2, v_3, v_4\} \\
\Delta_2 &= \{v_5, v_6, v_7, v_8, v_9, v_{10}\} \\
\Delta_3 &= \{v_{11}, v_{12}, v_{13}, v_{14}\} \\
\Delta_4 &= \{v_{15}\}
\end{aligned}$$

We begin by choosing a linear order $\lhd'$ on Δ_1 such that

$$v_1 \lhd' v_2 \lhd' v_3 \lhd' v_4. \tag{12}$$

Now we seek to bound the number of linear orders $\lhd''$ on Δ_2 such that $\lhd'$ and $\lhd''$ are compatible. By inspection, we find that (12) implies

$$v_5 \lhd'' v_6 \lhd'' \left\{ \begin{array}{c} v_7 \\ v_8 \end{array} \right\} \lhd'' v_9 \lhd'' v_{10}.$$

Given these inequalities, we have a choice of two distinct linear orders $\lhd''_1$ and $\lhd''_2$ on Δ_2, such that $v_7 \lhd''_1 v_8$ and $v_8 \lhd''_2 v_7$.

Next we find the sets $\eta_1, \eta_2, \eta_3, \eta_4$ containing $\zeta_1, \zeta_2, \zeta_3, \zeta_4$, respectively, for $v_1, \ldots, v_4 \in \Delta_1$, with respect to each of $\lhd''_1$ and $\lhd''_2$. We will begin with the former. By inspection, we find $v_1, v_2 < v_5$, and hence $\zeta_1, \zeta_2 = \eta_1, \eta_2 := \{0\}$. Further, $v_3 < v_6$, but v_3 is not necessarily less than v_5 under $<$, which implies $\zeta_3 \subset \eta_3 := \{0, 1\}$; finally, $v_4 < v_7$, but v_4 is not necessarily less than v_5 or v_6 under $<$, and so $\zeta_4 \subset \eta_4 := \{0, 1, 2\}$.

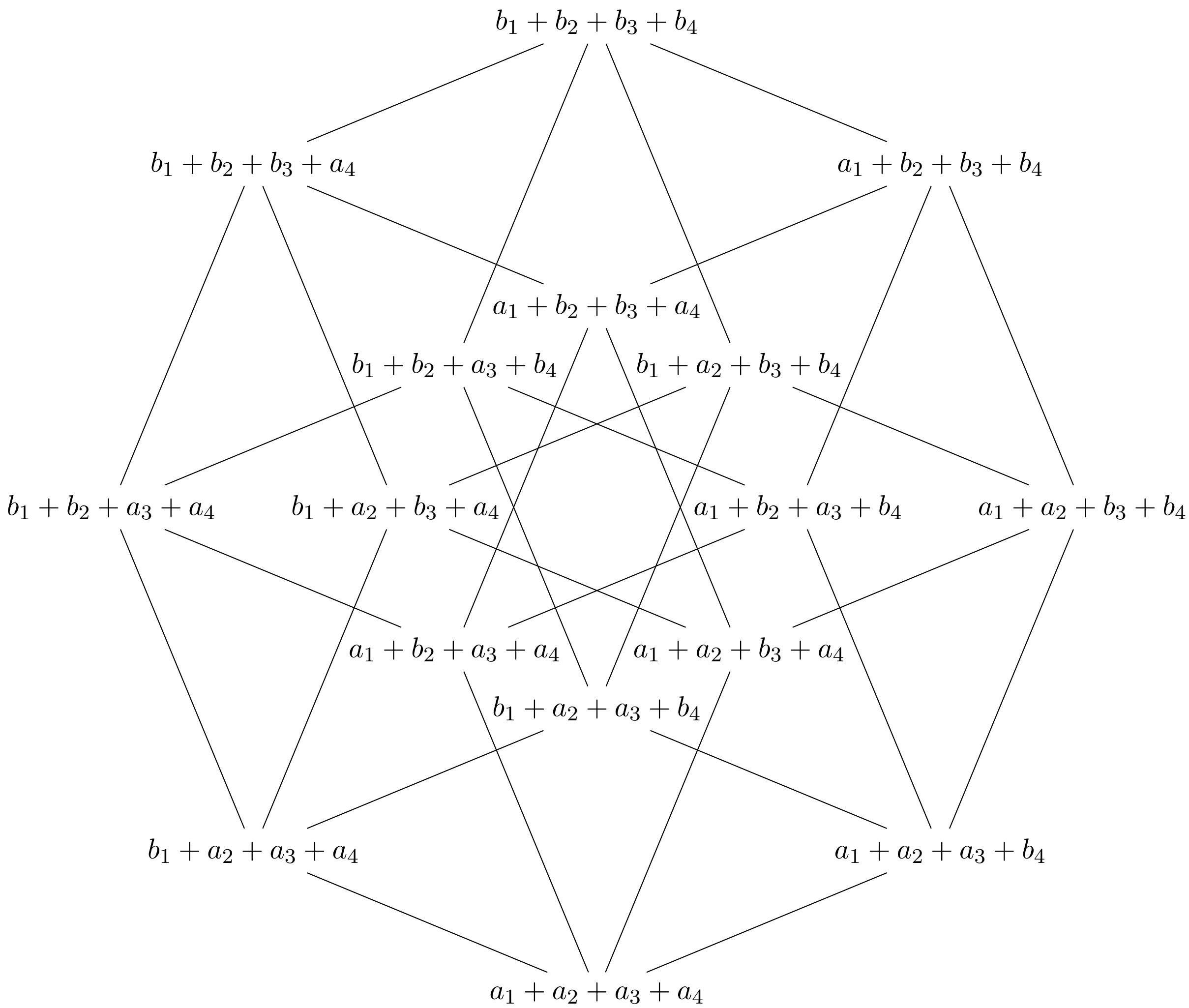


Figure 4: $(\mathcal{F}, <)$ with additive M and $k = 4$.

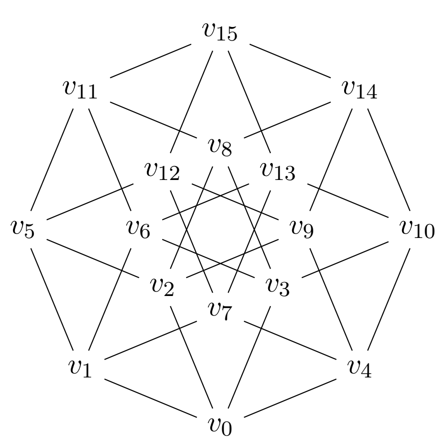


Figure 5: Shorthand for elements of $(\mathcal{F}, <)$ in the additive case with $k = 4$.

Now for $\lhd_2''$, many of the inequalities induced by $\lhd_1''$ remain valid in this case. In fact, the only change is that v_4 may now occupy the position above v_8, so that $\eta_4 = \{0, 1, 2, 3\}$, while η_1, η_2 and η_3 remain the same.

We now substitute these sets into (4). For $\lhd_1''$, we find that

$$\sum_{j_1 \in \{0\}} \sum_{j_2 \in h_2} \sum_{j_3 \in h_3} |h_4| = 5$$

and for $\lhd_2''$, we have

$$\sum_{j_1 \in \{0\}} \sum_{j_2 \in h_2} \sum_{j_3 \in h_3} |h_4| = 7$$

This gives the following twelve linear orders $\lhd = \lhd_i$, $i = 1, \ldots, 12$, on the union $\Delta_1 \cup \Delta_2$.

$$
\begin{gathered}
v_1 \lhd_1 v_2 \lhd_1 v_3 \lhd_1 v_4 \lhd_1 v_5 \lhd_1 v_6 \lhd_1 v_7 \lhd_1 v_8 \lhd_1 v_9 \lhd_1 v_{10} \\
v_1 \lhd_2 v_2 \lhd_2 v_3 \lhd_2 v_5 \lhd_2 v_4 \lhd_2 v_6 \lhd_2 v_7 \lhd_2 v_8 \lhd_2 v_9 \lhd_2 v_{10} \\
v_1 \lhd_3 v_2 \lhd_3 v_3 \lhd_3 v_5 \lhd_3 v_6 \lhd_3 v_4 \lhd_3 v_7 \lhd_3 v_8 \lhd_3 v_9 \lhd_3 v_{10} \\
v_1 \lhd_4 v_2 \lhd_4 v_5 \lhd_4 v_3 \lhd_4 v_4 \lhd_4 v_6 \lhd_4 v_7 \lhd_4 v_8 \lhd_4 v_9 \lhd_4 v_{10} \\
v_1 \lhd_5 v_2 \lhd_5 v_5 \lhd_5 v_3 \lhd_5 v_6 \lhd_5 v_4 \lhd_5 v_7 \lhd_5 v_8 \lhd_5 v_9 \lhd_5 v_{10} \\
v_1 \lhd_6 v_2 \lhd_6 v_3 \lhd_6 v_4 \lhd_6 v_5 \lhd_6 v_6 \lhd_6 v_8 \lhd_6 v_7 \lhd_6 v_9 \lhd_6 v_{10} \\
v_1 \lhd_7 v_2 \lhd_7 v_3 \lhd_7 v_5 \lhd_7 v_4 \lhd_7 v_6 \lhd_7 v_8 \lhd_7 v_7 \lhd_7 v_9 \lhd_7 v_{10} \\
v_1 \lhd_8 v_2 \lhd_8 v_3 \lhd_8 v_5 \lhd_8 v_6 \lhd_8 v_4 \lhd_8 v_8 \lhd_8 v_7 \lhd_8 v_9 \lhd_8 v_{10} \\
v_1 \lhd_9 v_2 \lhd_9 v_3 \lhd_9 v_5 \lhd_9 v_6 \lhd_9 v_8 \lhd_9 v_4 \lhd_9 v_7 \lhd_9 v_9 \lhd_9 v_{10} \\
v_1 \lhd_{10} v_2 \lhd_{10} v_5 \lhd_{10} v_3 \lhd_{10} v_4 \lhd_{10} v_6 \lhd_{10} v_8 \lhd_{10} v_7 \lhd_{10} v_9 \lhd_{10} v_{10} \\
v_1 \lhd_{11} v_2 \lhd_{11} v_5 \lhd_{11} v_3 \lhd_{11} v_6 \lhd_{11} v_4 \lhd_{11} v_8 \lhd_{11} v_7 \lhd_{11} v_9 \lhd_{11} v_{10} \\
v_1 \lhd_{12} v_2 \lhd_{12} v_5 \lhd_{12} v_3 \lhd_{12} v_6 \lhd_{12} v_8 \lhd_{12} v_4 \lhd_{12} v_7 \lhd_{12} v_9 \lhd_{12} v_{10}
\end{gathered}
$$

With these twelve choices we apply the algorithm to $\Delta_1 \cup \Delta_2$ and Δ_3. By Theorem 5.5 the linear order $\lhd'$ on Δ_1 implies a unique linear order $\lhd'''$ on Δ_3. In particular,

$$v_{11} \lhd''' v_{12} \lhd''' v_{13} \lhd''' v_{14}.$$

Now we must find η_i for each $u_i \in \Delta_1 \cup \Delta_2$ (here we will assume $u_i = v_i$ for $i = 1, \ldots, 10$). However, because this is an additive case, Theorem 5.4 simplifies this step significantly. For example, we observe that $v_1, v_2, v_3 < v_{11}$, and therefore $\eta_i = \{0\}$ for $u_i = v_1, v_2, v_3$. By Theorem 5.4, the positions of these three elements of Δ_1 in the linear order on $(\Delta_1 \cup \Delta_2)$ imply unique positions of their complements in the linear order on $(\Delta_1 \cup \Delta_2 \cup \Delta_3)$. We display the sets η_i in Table 1.

Finally, we substitute these sets into (4). The results are displayed in Table 2. We conclude that there exist at most 14 realizable linear extensions of $\mathcal{F}$ consistent with our choice of $\lhd'$ on Δ_1. Since $\lhd'$ is an arbitrary choice of one of 4! possible permutations, the total number of realizable linear extensions is at most $4! \cdot 14 = 336$.

7 Multiplicative example

In this example we consider a logic function M such that $M(c_1, c_2, c_3) = (c_1 + c_2)c_3$, which gives rise to the poset in Figure 2b. Compared with additive logic functions, this additive-multiplicative function is less restrictive, and the results of Section 5 do not hold in general.

$\lhd$	η_1	η_2	η_3	η_4	η_5	η_6	η_7	η_8	η_9	η_{10}
$\lhd_1$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
$\lhd_2$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{1\}$
$\lhd_3$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{1\}$	$\{1\}$
$\lhd_4$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{2\}$
$\lhd_5$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{1\}$	$\{2\}$
$\lhd_6$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
$\lhd_7$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{1\}$
$\lhd_8$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{1\}$	$\{1\}$
$\lhd_9$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0,1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$\lhd_{10}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{2\}$
$\lhd_{11}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{1\}$	$\{2\}$
$\lhd_{12}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0,1\}$	$\{1\}$	$\{1\}$	$\{2\}$

Table 1: η_i for $\Delta_1 \cup \Delta_2$ and Δ_3 in Example 6.2.

$\lhd$	$\lhd_1$	$\lhd_2$	$\lhd_3$	$\lhd_4$	$\lhd_5$	$\lhd_6$	$\lhd_7$	$\lhd_8$	$\lhd_9$	$\lhd_{10}$	$\lhd_{11}$	$\lhd_{12}$
No. of extensions	1	1	1	1	1	1	1	1	2	1	1	2

Table 2: Bounds on number of compatible linear extensions consistent with each order on $\Delta_1 \cup \Delta_2$ in Example 6.2, as computed by (4) with set η_1 and $h_2, h_3, \dots, h_{10}$.

We begin by following the additive examples and imposing an arbitrary linear order on the set of differences $\{(b_1 - a_1), (b_2 - a_2)\}$, since we have $\sigma_1 + \sigma_2$ as a factor of $M \circ \sigma$. We will choose

$$(b_1 - a_1) < (b_2 - a_2). \tag{13}$$

The blocks in $\mathcal{F}$ are as follows.

$$\begin{aligned}
\Delta_0 &= \{(a_1 + a_2)a_3\} \\
\Delta_1 &= \{(b_1 + a_2)a_3, (a_1 + b_2)a_3, (a_1 + a_2)b_3\} \\
\Delta_2 &= \{(b_1 + b_2)a_3, (b_1 + a_2)b_3, (a_1 + b_2)b_3\} \\
\Delta_3 &= \{(b_1 + b_2)b_3\}
\end{aligned}$$

First, we deal with the compatible linear orders $\lhd'$ and $\lhd''$ on Δ_1 and Δ_2 satisfying (13). We find that $(b_1 + a_2)a_3 \lhd' (a_1 + b_2)a_3$, and that none of our assumptions restrict $(a_1 + a_2)b_3$ from any of the three places under $\lhd'$. Similarly, $(b_1 + a_2)b_3 \lhd' (a_1 + b_2)b_3$, while $(b_1 + b_2)a_3$ may occupy any place under $\lhd''$. This yields the following linear orders $\lhd'$ on Δ_1 and $\lhd''$ on Δ_2 satisfying (13).

(A) $(b_1 + a_2)a_3 \lhd' (a_1 + b_2)a_3 \lhd' (a_1 + a_2)b_3$

(B) $(b_1 + a_2)a_3 \lhd' (a_1 + a_2)b_3 \lhd' (a_1 + b_2)a_3$

(C) $(a_1 + a_2)b_3 \lhd' (b_1 + a_2)a_3 \lhd' (a_1 + b_2)a_3$

(α) $(b_1 + b_2)a_3 \lhd'' (b_1 + a_2)b_3 \lhd'' (a_1 + b_2)b_3$

(β) $(b_1 + a_2)b_3 \lhd'' (b_1 + b_2)a_3 \lhd'' (a_1 + b_2)b_3$

(γ) $(b_1 + a_2)b_3 \lhd'' (a_1 + b_2)b_3 \lhd'' (b_1 + b_2)a_3$

Suppose $\lhd^*$ is a realizable linear extension of the partial order $\lhd$ on $\Delta_1 \cup \Delta_2$ formed from the disjoint union of $\lhd'$ and $\lhd''$. Then by realizability, if $(a_1 + b_2)a_3 \lhd^* (a_1 + a_2)b_3$, then because $0 < a_i < b_i$ for $i = 1, 2, 3$, we see $(b_1 - a_1)a_3 + (a_1 + b_2)a_3$ is less than $(b_1 - a_1)b_3 + (a_1 + a_2)b_3$ in $\mathbb{R}$. Simplifying gives $(b_1 + b_2)a_3 \lhd^* (b_1 + a_2)b_3$. Similarly, if $(b_1 + a_2)a_3 \lhd^* (a_1 + a_2)b_3$, then $(b_2 - a_2)a_3 + (b_1 + a_2)a_3$ is less than $(b_2 - a_2)b_3 + (a_1 + a_2)b_3$ in $\mathbb{R}$, and simplifying gives $(b_1 + b_2)a_3 \lhd^* (a_1 + b_2)b_3$. As a consequence, only (α) is compatible with (A), and only (α) and (β) are compatible with (B).

Beginning with the linear orders arising from (A) and (α), we find $(b_1 + a_2)a_3, (a_1 + b_2)a_3 < (b_1 + b_2)a_3$ and $(a_1 + a_2)b_3 < (b_1 + a_2)b_3$, but $(a_1 + a_2)b_3$ is not necessarily less than $(b_1 + b_2)a_3$ under $<$. Hence $\eta_1 = \eta_2 = \{0\}$ and $\eta_3 = \{0, 1\}$.

Next we consider the linear orders consistent with (B) and (α). Note that $(a_1 + b_2)a_3 < (b_1 + b_2)a_3$, so $\eta_1 = \eta_2 = \eta_3 = \{0\}$. For (B) and (β), we note that $(a_1 + a_2)b_3 < (b_1 + a_2)b_3$, so $\eta_1 = \eta_2 = \{0\}$. Since $(a_1 + b_2)a_3 < (b_1 + b_2)a_3$, but $(a_1 + b_2)a_3$ is not necessarily less than $(b_1 + a_2)b_3$ under $<$, $\eta_3 = \{0, 1\}$.

Finally, for (C) and (α) we find $\eta_1 = \eta_2 = \eta_3 = \{0\}$; for (β) we have $\eta_1 = \eta_2 = \{0\}$ and $\eta_3 = \{0, 1\}$; and for (γ) we again have $\eta_1 = \eta_2 = \{0\}$ and $\eta_3 = \{0, 1\}$.

We substitute each η_i into (4).

$$\sum_{j_1 \in \{0\}} \sum_{j_2 \in h_2} |h_3| = \begin{cases} 1 & \text{if } \eta_3 = \{0\} \\ 2 & \text{if } \eta_3 = \{0, 1\} \end{cases}$$

Because the pattern $\eta_1 = \eta_2 = \eta_3 = \{0\}$ occurs twice and $\eta_1 = \eta_2 = \{0\}, \eta_3 = \{0, 1\}$ four times, there are at most $2 \cdot 1 + 4 \cdot 2 = 10$ realizable linear extensions of $<$ consistent with (13). Identical considerations are valid when the inequality in (13) is reversed, and therefore there exist at most $2 \cdot 10 = 20$ realizable linear extensions.

8 Conclusion

We have introduced a construction and counting problem important to the dynamical classification of switching systems across global parameter space [5]. We have shown that the parameterization of a switching system gives rise to a natural partial order that is subject to algebraic constraints. The resulting poset has a special block structure that simplifies the construction of realizable linear extensions. The remaining open question is how to assess whether two linear orders on blocks can be merged in such a way that the algebraic constraints of the system are satisfied. We give further results for the special case of purely additive constraints, and we demonstrate these results on several examples.

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References

[1] G. Brightwell and P. Winkler, *Counting linear extensions*, Order, 8 (1991), pp. 225–242.

[2] E. R. Canfield and S. G. Williamson, *A loop-free algorithm for generating the linear extensions of a poset*, Order, 12 (1995), pp. 57–75.

[3] R. Casey, H. de Jong, and J. L. Gouzé, *Piecewise-linear models of genetic regulatory networks: equilibria and their stability*, J. Math. Biol., 56 (2006), pp. 27–56.

[4] M. Chaves, E. D. Sontag, and R. Albert, *Methods of robustness analysis for Boolean models of gene control networks*, IEE Proceedings-Systems Biology, 153 (2006), pp. 154–167.

[5] B. Cummins, T. Gedeon, S. Harker, K. Mischaikow, and K. Mok, *Combinatorial representation of parameter space for switching networks*, SIAM J. Appl. Dyn. Syst., 15 (2016), pp. 2176–2212.

[6] M. Davidich and S. Bornholdt, *From differential equations to Boolean networks: a case study in modeling regulatory networks.* manuscript (2008).

[7] H. de Jong, *Modeling and simulation of genetic regulatory systems: a literature review*, J. Comput. Biol., 9 (2002), pp. 67–103.

[8] H. de Jong, J. L. Gouzé, C. Hernandez, M. Page, T. Sari, and J. Geiselmann, *Qualitative simulation of genetic regulatory networks using piecewise-linear models*, Bull. Math. Biol., 66 (2004), pp. 301–340.

[9] R. Edwards, *Analysis of continuous-time switching networks*, Phys. D, 146 (2000), pp. 165–199.

[10] R. Edwards and L. Ironi, *Periodic solutions of gene networks with steep sigmoidal regulatory functions*, Phys. D, 282 (2014), pp. 1–15.

[11] L. Glass and S. A. Kauffman, *The logical analysis of continuous, non-linear biochemical control networks*, J. Theoret. Biol., 39 (1973), pp. 103–129.

[12] J. L. Gouzé and T. Sari, *A class of piecewise linear differential equations arising in biological models*, Dyn. Syst., 17 (2002), pp. 299–316.

[13] L. Ironi, L. Panzeri, E. Plahte, and V. Simoncini, *Dynamics of actively regulated gene networks*, Phys. D, 240 (2011), pp. 779–794.