Understanding the Eigenstructure of Various Triangles

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Abstract

We examine the eigenstructure of generalized isosceles triangles and explore the possibilities of analytic solutions to the general eigenvalue problem in other triangles. Starting with work based off of Brian McCartin's paper on equilateral triangles, we first explore the existence of analytic solutions within the space of all isosceles triangles. We find that this method only leads to consistent solutions in the equilateral case. Next, we develop criteria for the existence of complete solutions in other triangles. We find that complete solutions are guaranteed in the equilateral, right isosceles and 30-60-90 triangles. We then use a method developed by Milan Prager to formulate solutions in the right isosceles triangle through folding transformations of solutions in the square.

1 Introduction

Analyzing the eigenstructure of an equilateral triangle was studied by Lamé [3] and later addressed by Pinsky [8]. Pinsky's approach utilizes reflection operators and relies on a result from Arnol'd [1] to show that all the eigenfunctions are found. Seeing a gap in this literature regarding the eigenstructure of the equilateral triangle Brian J. McCartin in a series of papers provides an elementary treatment of the problem under Dirichlet [5], Neumann [4], and Robin [6] conditions. We extend his method to formulate the eigenstructures of other triangles. We examine, specifically in the isosceles case, under what conditions analytic solutions exist. Next we focus on the homogeneous Dirichlet problem in three different cases: the equilateral triangle, the 30-60-90 triangle and the right isosceles triangle.

To develop solutions for the isosceles triangles, we modify a triangular coordinate system presented in [5] and then develop an orthogonal coordinate system in order to generate solutions using separation of variables. This method leads to consistent solutions

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Financial support for this project provided by the NSF's Mentoring Through Critical Transition Points (MCTP) Grant, No DMS-0602284.
Acknowledgments: We would like to thank our advisors Jim Curry and Anne Dougherty for their help, as well as the reviewers for their insightful comments and pointing out references [1] and [7].

only in the case of the equilateral triangle. However, in the coordinate system that we have developed, we cannot derive solutions in the right isosceles triangle.

Due to the inconsistencies that were found for generalized isosceles triangles, we explore criteria under which we can warranty the existence of complete symmetric and anti-symmetric solutions for the triangular domain. We consider under what cases we can tile the plane with a triangular region solely through anti-symmetric reflections in such a way that the nodal lines of the extensions line up properly. This analysis leads to solutions only in the cases of equilateral, 30-60-90, and isosceles right triangles.

Because of the aforementioned issues with our extension of the method presented by McCartin, we examine solutions for the eigenvalue problem in a 30-60-90 triangle using a method developed by Milan Prager in [9] and extended upon in [10]. In this method, Prager uses "folding" and "prolongation" transformations of solutions in a rectangle to obtain solutions in the 30-60-90 triangle. We then adapt this method to right isosceles triangles through similar transformations of a square.

2 McCartin's Method: Redefinition of Eigenvalue Problem

The method developed in [5] we will refer to as McCartins Method. McCartin's Method requires a different look at the traditional eigenvalue problem in $\mathbb{R}^2$. Instead of defining the boundaries under Cartesian coordinates, he develops a triangular coordinate system using the altitudes of an equilateral triangle. We emulate this process for a general isosceles triangle.

2.1 The Development of a Coordinate System

To solve the eigenvalue problem in an equilateral triangle, we need a coordinate system that allows the boundary conditions to be applied along a constant boundary. We start with a coordinate system presented by McCartin [5] and modify that set of coordinates to be valid in any isosceles triangle. We define our triangle as indicated in Figure 1 where two sides are of length s and the third side is of length αs, where $\alpha \in (0, 2)$. Note that the cases $\alpha = 1$ and $\alpha = \sqrt{2}$ correspond to an equilateral triangle and a right isosceles triangle, respectively. We then define the coordinate transformation:

$$u = \left(\frac{\alpha}{2}\right)^2 s \left(1 - \frac{\alpha^2}{4}\right)^{-\frac{1}{2}} - y, \tag{1}$$

$$v = \sqrt{1 - \frac{\alpha^2}{4}} \left(x - \frac{\alpha}{2}s\right) + \frac{\alpha}{2}\left(y - \left(\frac{\alpha}{2}\right)^2 s \left(1 - \frac{\alpha^2}{4}\right)^{-\frac{1}{2}}\right), \tag{2}$$

$$w = \sqrt{1 - \frac{\alpha^2}{4}} \left(\frac{\alpha}{2}s - x\right) + \frac{\alpha}{2}\left(y - \left(\frac{\alpha}{2}\right)^2 s \left(1 - \frac{\alpha^2}{4}\right)^{-\frac{1}{2}}\right). \tag{3}$$

From those equations it can be seen that the u-axis bisects the triangle vertically. The v-axis travels from the xy-origin to intersect the opposite side of the triangle at a right

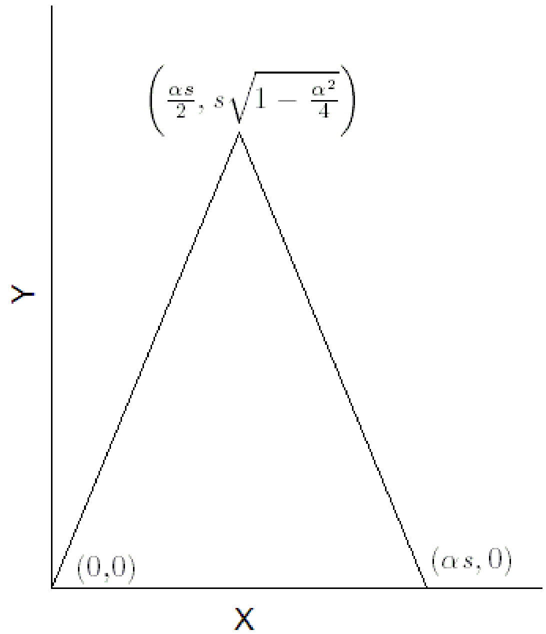


Figure 1: Isosceles Triangle

angle. The w-axis behaves similarly to the v-axis, but travels from the corner at $(\alpha s, 0)$ to form a right angle with the opposite side. These axes can be seen in Figure 2. All three of these new axes will intersect at a single point, we define that point to be $u = v = w = 0$ and call it the uvw-origin. A point in the triangle is defined in (u, v, w) by finding the orthogonal projection of the point onto the three new axes. When using the (u, v, w) location with distances from the point $(0, 0, 0)$, we define the positive direction for the axes as the end that intersects the side of the triangle. In the standard coordinate system, we found the origin point to be

$$(x, y) = \left(\frac{\alpha}{2}s, \left(\frac{\alpha}{2}\right)^2 s\left(1 - \frac{\alpha^2}{4}\right)^{-\frac{1}{2}}\right). \tag{4}$$

We find the x-coordinate by simply moving down half of the base side length, and the y-coordinate through an angle given by:

$$\theta_2 = \arcsin\left(\frac{\alpha}{2}\right). \tag{5}$$

θ_2 exists between the base of the triangle and either the v- or w-axis. The right triangle formed by the right half of the isosceles triangle is similar to the triangle formed by the w-axis, the base, and the left side of the triangle. These two triangles are related by a factor of α; the angle θ_2 is consequently half of the top angle of the original isosceles triangle. Now we can find the height of the origin because we can form a right triangle with the negative part of the v axis, the positive part of u and the base of the isosceles triangle. We get the following expression for the y-coordinate of the uvw-origin:

$$\frac{2\tan(\theta_2)}{\alpha s}, \tag{6}$$

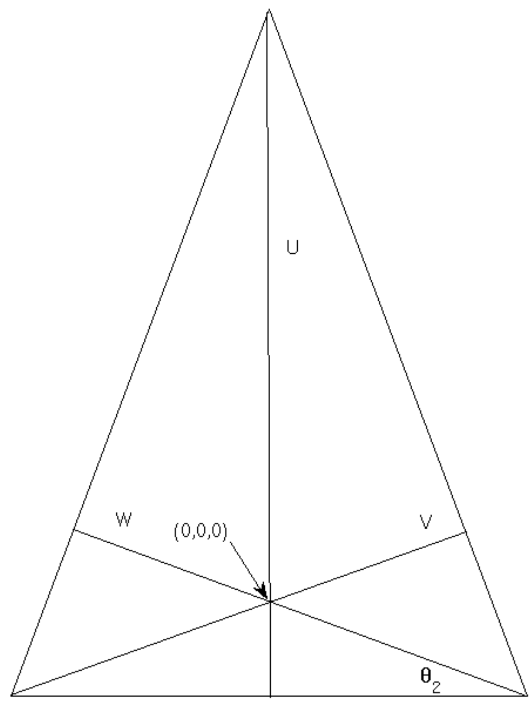


Figure 2: Isosceles Triangle with u,v,w axis

which simplifies to (4). Equations (1), (2), and (3) are derived by orthogonally projecting a point onto each of the three axis, and then adjusting for the points location relative to the uvw-origin. We also derive the following relationship for (u, v, w):

$$\alpha u + v + w = 0. \tag{7}$$

To establish domains for u, v and w, we need to know the length of each axis inside the triangular domain. The length of the u-axis, which we will call $|\vec{u}|$, is simply the height of the triangle:

$$|\vec{u}| = s\sqrt{1 - \frac{\alpha^2}{4}}. \tag{8}$$

To obtain the length of the w-axis, we consider the triangle formed by the axis in question and the two congruent sides of the isosceles triangle. Using Pythagorean Theorem, we calculate the length of the axis to be:

$$|\vec{w}| = \alpha s\sqrt{1 - \frac{\alpha^2}{4}}. \tag{9}$$

The v-axis is congruent to the w-axis, so from (8) and (9), we obtain the relation $|\vec{w}| = |\vec{v}| = \alpha\,|\vec{u}|$. The lengths of the axes are useful because they are also the interval lengths for our domains on u, v, and w, and by finding one boundary for the interval, we immediately know the other. From the location of the origin in the uvw-plane and the triangle's vertices, we obtain the following bounds for the coordinates:

$$\begin{aligned}
u_{max} &= \frac{\alpha}{2}s\tan(\theta_2), \\
u_{min} &= -s\sqrt{1-\frac{\alpha^2}{4}} + \frac{\alpha}{2}s\tan(\theta_2), \\
w_{max} &= \alpha s\sqrt{1-\frac{\alpha^2}{4}} - \frac{\alpha s}{2\cos(\theta_2)}, \\
w_{min} &= -\frac{\alpha s}{2\cos(\theta_2)}, \\
v_{min} &= w_{min}, \\
v_{max} &= w_{max}.
\end{aligned}$$

Essentially, for every point inside the triangle, $u \in [u_{min}, u_{max}]$, $v \in [v_{min}, v_{max}]$, and $w \in [w_{min}, w_{max}]$ must all be satisfied.

2.2 Transformed Laplacian Equation

Using the coordinate system presented above, we use separation of variables as our primary solution method starting with:

$$\nabla^2 T - K^2 T = 0. \tag{10}$$

We assume that T has the form $f(u) \cdot g(v-w)$. We use $(v-w)$ because it gives us an orthogonal coordinate system. First, we need to find our new Laplacian for this space. We apply the change of variables $\sigma = u$ and $\eta = (v-w)$, placing those variables into the Laplacian to derive:

$$f(\eta) = f\left(2\cos(\theta_2)\left(x - \frac{\alpha}{s}\right)\right), \tag{11}$$

$$f(\sigma) = f\left(\frac{\alpha}{2}s\tan(\theta_2) - y\right). \tag{12}$$

Using the Chain Rule we get the following transformation of the Laplacian from x- and y-coordinates to σ- and η-coordinates:

$$\nabla^2 T(x,y) = \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 T}{\partial \sigma^2} + (4-\alpha^2)\frac{\partial^2 T}{\partial \eta^2} = \nabla^2 T(\sigma,\eta). \tag{13}$$

Now that we have an orthogonal coordinate system we solve our problem through the use of separation of variables. We assume a solution of the form $T = f(\sigma)g(\eta)$, substituting this into (10) and using (13) to obtain:

$$\frac{1}{f(\sigma)}\frac{\partial^2 f}{\partial \sigma^2}(\sigma) + (4-\alpha^2)\frac{1}{g(\eta)}\frac{\partial^2 g}{\partial \eta^2}(\eta) + K^2 = 0. \tag{14}$$

The first two parts of (14) are only in terms of σ and η respectively. K^2 is a constant, so we know that the only way for the sum of the three parts to be zero is if the individual parts equal constants:

$$\frac{1}{f(\sigma)}\frac{\partial^2 f}{\partial \sigma^2}(\sigma) = -A^2, \tag{15}$$

$$\frac{1}{g(\eta)}\frac{\partial^2 g}{\partial \eta^2}(\eta) = -B^2. \tag{16}$$

This gives us the relation $K^2 = A^2 + (4-\alpha^2)B^2$. We can now solve each one-dimensional eigenvalue problem:

$$\frac{\partial^2 f}{\partial \sigma^2}(\sigma) + A^2 f(\sigma) = 0, \tag{17}$$

$$\frac{\partial^2 g}{\partial \eta^2}(\eta) + B^2 g(\eta) = 0. \tag{18}$$

2.3 Symmetric Solutions for a General Isosceles Triangle

Following McCartin's method, we explore two types of solutions to (10): symmetric and anti-symmetric. We define $T_s(u,v,w)$ to be the solution symmetric about the u-axis and $T_a(u,v,w)$ to be the solution anti-symmetric about the u-axis, according to these equations:

$$T_s(u,v,w) = \frac{T(u,v,w)+T(u,w,v)}{2}, \tag{19}$$

$$T_a(u,v,w) = \frac{T(u,v,w)-T(u,w,v)}{2}. \tag{20}$$

For now, we only consider symmetric solutions. The anti-symmetric solutions follow the same method, and simply involve sine functions in one of the coordinate directions.

2.3.1 Homogeneous Boundary Conditions in u

Since we have homogeneous boundary conditions, T_s must vanish when $u = u_{min}$ and $u = u_{max}$ while also satisfying the Laplacian equations. Based on these homogeneous conditions we look for solutions $T_s = f(u)g(v-w)$ that are a product of trigonometric functions. To make it symmetric about the u-axis, $g(v-w)$ must be the cosine function. Similarly for the anti-symmetric case we would need sine functions. To force homogeneous conditions on the boundaries for u, $f(u)$ must be the sine function centered around $u = u_{min}$. Hence,

$$T_s = \sin(A(u-u_{min}))\cos(B(v-w)), \tag{21}$$

which vanishes when $u = u_{min}$. In order for the function to vanish along $u = u_{max}$, we substitute it into the equation to find:

$$\begin{aligned} T_s &= \sin(A(u_{max}-u_{min}))\cos(B(v-w)) = 0, \\ &\to 0 = \sin(A(u_{max}-u_{min})), \\ &\to \pi l = (A(u_{max}-u_{min})), \\ &\to A = \frac{\pi l}{s\sqrt{1-\frac{\alpha^2}{4}}}, \end{aligned} \tag{22}$$

where l is some integer. As a result, we can express the symmetric solution as:

$$T_s = \sin\left[\frac{\pi l}{s\sqrt{1-\frac{\alpha^2}{4}}}(u-u_{min})\right]\cos\left[B(v-w)\right]. \tag{23}$$

2.3.2 The Other Homogeneous Boundary Conditions

Now we enforce the other boundary conditions, in which T_s must vanish along the lines $v = v_{max}$ and $w = w_{max}$. However, the symmetry of the isosceles triangle and the evenness of the cosine function lets us conclude that satisfying the homogeneous conditions on one side, say $v = v_{max}$, would also satisfy the conditions on the other side. Hence, we only focus on the condition $v = v_{max}$:

$$\begin{aligned} v = v_{max} \quad &\rightarrow \quad \alpha u + v + w = \alpha u + v_{max} + w = 0, \\ &\rightarrow \quad -w = \alpha u + v_{max}, \\ &\rightarrow \quad v - w = (v_{max}) + (\alpha u + v_{max}), \\ &\rightarrow \quad v - w = \alpha u + 2v_{max}. \end{aligned} \tag{24}$$

The above equation describes the value of $(v - w)$ along $v = v_{max}$ boundary in terms of the independent variable u, which leads to interesting simplifications.

However, a few expressions need to be rewritten first. Since $2\sin(\theta_2) = \alpha$ also implies that $2\cos(\theta_2) = \sqrt{4-\alpha^2}$ and, by the definition of the tangent function, $\tan(\theta_2) = \frac{\alpha}{\sqrt{4-\alpha^2}}$, we can rewrite u_{min} as:

$$\begin{aligned} u_{min} &= -\frac{s}{2}\sqrt{4-\alpha^2} + \frac{\alpha^2 s}{2\sqrt{4-\alpha^2}}, \\ &= -s\left(\frac{\sqrt{4-\alpha^2}}{2} - \frac{\alpha^2}{2\sqrt{4-\alpha^2}}\right), \\ &= -s\left(\frac{(4-\alpha^2)-\alpha^2}{2\sqrt{4-\alpha^2}}\right), \\ &= -s\left(\frac{2-\alpha^2}{\sqrt{4-\alpha^2}}\right). \end{aligned} \tag{25}$$

Furthermore, similar substitutions can be made to v_{max}, leading to:

$$\begin{aligned} v_{max} &= \frac{\alpha s}{2}\sqrt{4-\alpha^2} - \frac{\alpha s}{\sqrt{4-\alpha^2}}, \\ &= \alpha s\left(\frac{\sqrt{4-\alpha^2}}{2} - \frac{1}{\sqrt{4-\alpha^2}}\right), \\ &= \alpha s\left(\frac{(4-\alpha^2)-2}{2\sqrt{4-\alpha^2}}\right), \\ &= \alpha s\left(\frac{2-\alpha^2}{2\sqrt{4-\alpha^2}}\right), \\ &= -\frac{\alpha}{2}u_{min}. \end{aligned} \tag{26}$$

The final expression in (26) is the most intriguing because it directly relates u_{min} and v_{max} with a linear function based on α. To check the veracity of this relation, one can substitute the α-value for an equilateral triangle ($\alpha = 1$) and confirm that the maximum value for v is indeed equal to the minimum of u multiplied by the quantity $-\frac{\alpha}{2}$.

More importantly, the expression in (26) allows us to simplify $(v - w)$ along the $v = v_{max}$ boundary found in Equation (24):

$$\begin{aligned} v - w &= \alpha u + 2v_{max}, \\ &= \alpha u + 2\left(-\frac{\alpha}{2}u_{min}\right), \\ &= \alpha(u - u_{min}). \end{aligned} \tag{27}$$

This allows us to express the symmetric solution in (23) along this boundary all in terms of u:

$$T_s = \sin\left[\frac{\pi l}{s\sqrt{1-\frac{\alpha^2}{4}}}(u - u_{min})\right]\cos\left[B_1\alpha(u - u_{min})\right]. \tag{28}$$

However, Equation (28) does not solve the boundary conditions; it only expresses the general form of the solution along the line $v = v_{max}$. No value of B_1 would lead to T_s vanishing along this line.

We know that Equation (28) solves the original boundary condition, so we can use a linear combination of these solutions to create a symmetric solution that vanishes along the necessary boundary. Of course, we need to create unique expressions of A and B for each added solution, but these expressions will be very similar in structure and still have to satisfy $K^2 = A^2 + (4 - \alpha^2)B^2$.

First, we use two versions of Equation (23) and derive the following expression for the solution along the boundary $v = v_{max}$:

$$\begin{aligned} T_s = \quad & \sin\left[\frac{\pi l}{s\sqrt{1-\frac{\alpha^2}{4}}}(u - u_{min})\right]\cos\left[B_1\alpha(u - u_{min})\right] \\ & + \sin\left[\frac{\pi m}{s\sqrt{1-\frac{\alpha^2}{4}}}(u - u_{min})\right]\cos\left[B_2\alpha(u - u_{min})\right]. \end{aligned} \tag{29}$$

Using the trigonometric identity

$$\sin x \cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y)), \tag{30}$$

Equation (29) can be rewritten as the following:

$$\begin{aligned} T_s = \quad & \frac{1}{2}\left\{\sin\left[\left(\frac{\pi l}{s\sqrt{1-\frac{\alpha^2}{4}}} + B_1\alpha\right)(u - u_{min})\right] + \sin\left[\left(\frac{\pi l}{s\sqrt{1-\frac{\alpha^2}{4}}} - B_1\alpha\right)(u - u_{min})\right]\right\} \\ + \quad & \frac{1}{2}\left\{\sin\left[\left(\frac{\pi m}{s\sqrt{1-\frac{\alpha^2}{4}}} + B_2\alpha\right)(u - u_{min})\right] + \sin\left[\left(\frac{\pi m}{s\sqrt{1-\frac{\alpha^2}{4}}} - B_2\alpha\right)(u - u_{min})\right]\right\}. \end{aligned} \tag{31}$$

In order for this new version of the symmetric solution to vanish for all values of u, one of the sine functions involving B_1 must equal the negative of a sine function involving B_2. It should be noted that if both sine functions involving B_1 canceled, then Equation (31) effectively reduces to the single solution form found in (28), which does not vanish. Hence, we need only to consider the pairings of B_1 functions and B_2 functions.

Two distinct systems of equations arise when requiring that T_s vanish. One system is:

$$\begin{aligned} \frac{\pi l}{s\sqrt{1-\frac{\alpha^2}{4}}} + B_1\alpha &= -\left(\frac{\pi m}{s\sqrt{1-\frac{\alpha^2}{4}}} + B_2\alpha\right), \\ \frac{\pi l}{s\sqrt{1-\frac{\alpha^2}{4}}} - B_1\alpha &= -\left(\frac{\pi m}{s\sqrt{1-\frac{\alpha^2}{4}}} - B_2\alpha\right), \end{aligned} \tag{32}$$

while the other system could look like this:

$$\begin{aligned} \frac{\pi l}{s\sqrt{1-\frac{\alpha^2}{4}}} - B_1\alpha &= -\left(\frac{\pi m}{s\sqrt{1-\frac{\alpha^2}{4}}} + B_2\alpha\right), \\ \frac{\pi l}{s\sqrt{1-\frac{\alpha^2}{4}}} + B_1\alpha &= -\left(\frac{\pi m}{s\sqrt{1-\frac{\alpha^2}{4}}} - B_2\alpha\right). \end{aligned} \tag{33}$$

Regardless of which system one chooses, the solutions are essentially the same: $l = -m$ and $|B_1| = |B_2|$, where B_1 and B_2 have opposite signs to solve the system in (32) or have the same signs to solve the system in (33). However, when either solution is substituted back into Equation (29), we obtain $T_s = 0$, the trivial solution. When reaching this step in his own paper, McCartin does not directly address the two possible systems of equations. However, neither system leads to any useful solutions, so his not mentioning the multiple systems here is understandable.

Hence, we consider the symmetric solution involving a third version of the expression in (23). We find values of B_1, B_2, and B_3 that satisfy the boundary conditions along $v = v_{max}$:

$$\begin{aligned} T_s = \quad & \sin\left[\frac{\pi l}{s\sqrt{1-\frac{\alpha^2}{4}}}(u-u_{min})\right]\cos\left[B_1\alpha(u-u_{min})\right] \\ + \quad & \sin\left[\frac{\pi m}{s\sqrt{1-\frac{\alpha^2}{4}}}(u-u_{min})\right]\cos\left[B_2\alpha(u-u_{min})\right] \\ + \quad & \sin\left[\frac{\pi n}{s\sqrt{1-\frac{\alpha^2}{4}}}(u-u_{min})\right]\cos\left[B_3\alpha(u-u_{min})\right], \end{aligned} \tag{34}$$

with the condition:

$$
\begin{aligned}
K^2 &= \left(\frac{\pi l}{s\sqrt{1-\frac{\alpha^2}{4}}}\right)^2 + (4-\alpha^2)B_1^2, \\
&= \left(\frac{\pi m}{s\sqrt{1-\frac{\alpha^2}{4}}}\right)^2 + (4-\alpha^2)B_2^2, \\
&= \left(\frac{\pi n}{s\sqrt{1-\frac{\alpha^2}{4}}}\right)^2 + (4-\alpha^2)B_3^2.
\end{aligned} \tag{35}
$$

Again, we use the identity in Equation (30) to rewrite (34) as:

$$
\begin{aligned}
T_s = \quad & \frac{1}{2}\left\{\sin\left[\left(\frac{\pi l}{s\sqrt{1-\frac{\alpha^2}{4}}}+B_1\alpha\right)(u-u_{min})\right]+\sin\left[\left(\frac{\pi l}{s\sqrt{1-\frac{\alpha^2}{4}}}-B_1\alpha\right)(u-u_{min})\right]\right\} \\
+ \quad & \frac{1}{2}\left\{\sin\left[\left(\frac{\pi m}{s\sqrt{1-\frac{\alpha^2}{4}}}+B_2\alpha\right)(u-u_{min})\right]+\sin\left[\left(\frac{\pi m}{s\sqrt{1-\frac{\alpha^2}{4}}}-B_2\alpha\right)(u-u_{min})\right]\right\} \\
+ \quad & \frac{1}{2}\left\{\sin\left[\left(\frac{\pi n}{s\sqrt{1-\frac{\alpha^2}{4}}}+B_3\alpha\right)(u-u_{min})\right]+\sin\left[\left(\frac{\pi n}{s\sqrt{1-\frac{\alpha^2}{4}}}-B_3\alpha\right)(u-u_{min})\right]\right\}.
\end{aligned} \tag{36}
$$

To make Equation (36) vanish for all values of u, we create a system of three equations obtained by matching one sine function of B_1 to the negative of a function of B_2, the other function of B_1 to the negative of a function of B_3, and the remaining function of B_2 to the negative of the remaining function of B_3. Any other matching scheme would reduce the problem to (28) or (29), which we have already shown leads to trivial or nonexistent solutions.

However, this scheme actually leads to eight different systems of equations that should lead to essentially the same solutions, with the possibility of having some B-value solutions being negative in some systems but positive in others. We will not list all eight systems and instead present one possible system from which we can solve for B_1, B_2, and

B_3:

$$\begin{aligned}
\frac{\pi l}{s\sqrt{1-\frac{\alpha^2}{4}}} + B_1\alpha &= -\left(\frac{\pi m}{s\sqrt{1-\frac{\alpha^2}{4}}} - B_2\alpha\right), \\
\frac{\pi m}{s\sqrt{1-\frac{\alpha^2}{4}}} + B_2\alpha &= -\left(\frac{\pi n}{s\sqrt{1-\frac{\alpha^2}{4}}} - B_3\alpha\right), \\
\frac{\pi n}{s\sqrt{1-\frac{\alpha^2}{4}}} + B_3\alpha &= -\left(\frac{\pi l}{s\sqrt{1-\frac{\alpha^2}{4}}} - B_1\alpha\right).
\end{aligned} \tag{37}$$

Adding these three equations together and grouping terms yields the following solvability condition:

$$l + m + n = 0. \tag{38}$$

The equation in (38) allows us to eliminate one of these variables by letting $l = -(m+n)$, for example, and writing the symmetric solution solely in terms of m and n.

It should also be noted that the solvability condition in Equation (38) does not depend on α or any other geometric constant, meaning that this condition does not come from the geometry of the coordinate system.

Returning to the task of solving the system in (37) for B_1, B_2, and B_3, we notice that any attempt to solve it would always reduce to the solvability condition. However, from the equations in (35), we can determine more equations involving B_1, B_2, and B_3 by simple algebra. Our complete system of equations becomes the following:

$$\begin{aligned}
B_1 - B_2 &= \frac{2\pi n}{\alpha s\sqrt{4-\alpha^2}}, \\
B_2 - B_3 &= \frac{2\pi l}{\alpha s\sqrt{4-\alpha^2}}, \\
B_3 - B_1 &= \frac{2\pi m}{\alpha s\sqrt{4-\alpha^2}}, \\
B_1 + B_2 &= \left(\frac{2\pi\alpha}{s(4-\alpha^2)^{3/2}}\right)(l-m), \\
B_2 + B_3 &= \left(\frac{2\pi\alpha}{s(4-\alpha^2)^{3/2}}\right)(m-n), \\
B_3 + B_1 &= \left(\frac{2\pi\alpha}{s(4-\alpha^2)^{3/2}}\right)(n-l).
\end{aligned} \tag{39}$$

This is where the process appears to break down. We have six equations for three unknown parameters, making this an over-determined system that is ultimately unsolvable. McCartin appears to avoid this issue in [5] by simply selecting two of the equations that have the same pair of unknown parameters, such as $(B_1 - B_2)$ and $(B_1 + B_2)$, and solving for them. As we will show, this reduced system yields consistent expressions for the B values in the equilateral triangle, so McCartin's choice does not invalidate his solutions. However, when we emulate this procedure for our generalized isosceles triangle, this process leads to inconsistent solutions.

To see why these solutions are inconsistent, we show what solutions can be obtained, first adding or subtracting the equations for $(B_1 - B_2)$ and $(B_1 + B_2)$ to find values for B_1 and B_2:

$$(B_1 + B_2) + (B_1 - B_2) = 2B_1 = \frac{2\pi\alpha(l-m)}{s(4-\alpha^2)^{3/2}} + \frac{2\pi n}{s\alpha\sqrt{4-\alpha^2}},$$
$$(B_1 + B_2) - (B_1 - B_2) = 2B_2 = \frac{2\pi\alpha(l-m)}{s(4-\alpha^2)^{3/2}} - \frac{2\pi n}{s\alpha\sqrt{4-\alpha^2}}.$$

Dividing by 2 and factoring yields:

$$B_1 = \frac{\pi}{s\sqrt{4-\alpha^2}}\left(\frac{\alpha}{4-\alpha^2}(l-m) + \frac{n}{\alpha}\right), \tag{40}$$

$$B_2 = \frac{\pi}{s\sqrt{4-\alpha^2}}\left(\frac{\alpha}{4-\alpha^2}(l-m) - \frac{n}{\alpha}\right). \tag{41}$$

Applying this process to the equations for $(B_3 - B_1)$ and $(B_3 + B_1)$ yields the following solutions for B_1 and B_3:

$$B_1 = \frac{\pi}{s\sqrt{4-\alpha^2}}\left(\frac{\alpha}{4-\alpha^2}(n-l) - \frac{m}{\alpha}\right), \tag{42}$$

$$B_3 = \frac{\pi}{s\sqrt{4-\alpha^2}}\left(\frac{\alpha}{4-\alpha^2}(n-l) + \frac{m}{\alpha}\right), \tag{43}$$

and again to equations for $(B_2 - B_3)$ and $(B_2 + B_3)$ yields the following solutions for B_2 and B_3:

$$B_2 = \frac{\pi}{s\sqrt{4-\alpha^2}}\left(\frac{\alpha}{4-\alpha^2}(m-n) + \frac{l}{\alpha}\right), \tag{44}$$

$$B_3 = \frac{\pi}{s\sqrt{4-\alpha^2}}\left(\frac{\alpha}{4-\alpha^2}(m-n) - \frac{l}{\alpha}\right). \tag{45}$$

For each of the unknown eigenvalues, we have two possible expressions which do not appear to match each other. This result leads us to a crucial question: are these two expressions equal to each other?

2.4 Triangles with Consistent Solutions

From the previous section, we found that there are two possible expressions for B_1 based on which equations we choose to solve. However, it is possible that there exists values of α that make both expressions consistent for all triplets (l, m, n) that satisfy the solvability condition in (38). To find such values, we subtract Equation (42) from Equation (40) and solve for α:

$$\frac{\pi}{s\sqrt{4-\alpha^2}}\left(\left(\frac{\alpha}{4-\alpha^2}(l-m) + \frac{n}{\alpha}\right) - \left(\frac{\alpha}{4-\alpha^2}(n-l) - \frac{m}{\alpha}\right)\right) = 0,$$

$$
\begin{aligned}
\frac{\alpha}{4-\alpha^2}((l-m)-(n-l)) &= \frac{-1}{\alpha}(n-(-m)),\\
\frac{\alpha}{4-\alpha^2}(2l-(m+n)) &= \frac{-1}{\alpha}(m+n).
\end{aligned}
$$

Applying the solvability condition in (38) yields:

$$
\begin{aligned}
\frac{3\alpha l}{4-\alpha^2} &= \frac{l}{\alpha},\\
3\alpha^2 l &= (4-\alpha^2)l,\\
4\alpha^2 l &= 4l,\\
\alpha^2 &= 1,\\
\alpha &= 1.
\end{aligned} \tag{46}
$$

The same results in (46) can be obtained for B_2 and B_3, meaning that α must be equal to one. Thus, the only triangle that yields consistent symmetric solutions for all B parameters is the equilateral triangle, and when we substitute ($\alpha = 1$), we obtain the same results as in [5].

3 Considered Methods for Non-Equilateral Triangles

Even though our analysis shows that McCartin's method only appears to apply to equilateral triangles, the real purpose of our investigation was to find practical methods of solving the Helmholtz equation in any triangle. Intuitively, we should be able to find solutions in more triangles than equilateral triangles, even if we have to restrict our domain to just the triangle itself instead of $\mathbb{R}^2$. In fact, McCartin even shows in [5] how purely anti-symmetric solutions in an equilateral triangle form a complete set of solutions in a 30-60-90 triangle. The limitations of his method does not necessarily preclude the existence of solutions in other types of polygons, only the existence of solutions of the form in (21). As a result, we explore other approaches to this problem, allowing for the existence of solution sets with alternative forms.

3.1 Criteria for the Existence of Complete Solutions

For us, a complete set of solutions is defined as a set of solutions solving Laplace's equation in the triangle that can create a Fourier series for any function defined on $\mathbb{R}^2$, as in, the solutions can be individually extended outside of the triangle. For McCartin, he assumes solutions that are either symmetric or anti-symmetric with respect to an altitude of the equilateral triangle and uses Lamé's Theorem to show how any function defined in the equilateral triangle can be expressed as a linear combination of these solution sets. He then proved that solutions of these types can be extended to domains outside of the equilateral triangle. The intriguing part about this second part is that he did not need to know the actual equation in order to prove extendibility. In fact, a graphical proof will suffice to show if symmetric or anti-symmetric solutions can be extended outside of the triangle, provided that the triangle meets meets certain criteria.

The first criteria is the ability to tile the entire plane, but only in a specific manner. While it is possible to tile the plane using any triangle (reflections about the midpoint of any side yields a parallelogram, which tiles the plane with translations), extensions of the solutions in the triangle can only be made through anti-symmetric reflections over an entire side of the triangle, not through rotation, translation, or reflection about a single point. The reason behind this requirement is to preserve the continuity of the function and its derivatives, which we can only guarantee for solutions with homogeneous boundary conditions if we use anti-symmetric reflections. (It turns out, with Neumann conditions, we must use symmetric reflections, but that it outside of our current field of inquiry.)

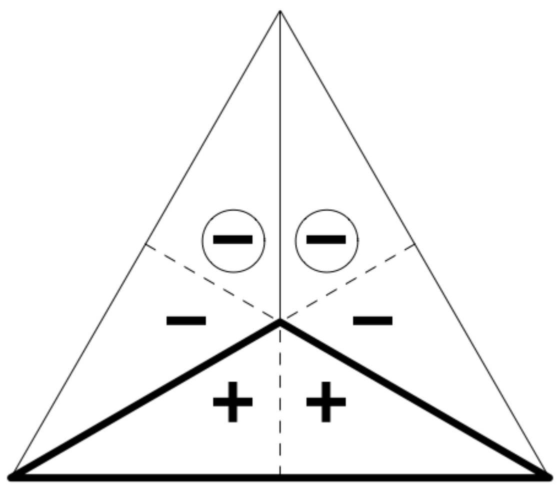


Figure 3: Extensions of Symmetric Solutions of the 30-30-120 Triangle

Regardless of the motivation, this restriction severely reduces the number of triangles that we need to consider. To find such a triangle, we start with one of the finite number of polygons known to tile the plane through reflections and instead tile that area with smaller triangles. For example, an equilateral triangle tiles the plane through reflections, and since two congruent 30-60-90 triangles form an equilateral triangle through reflections, we can conclude that 30-60-90 triangle can tile the plane. Similarly, a square can easily be shown to tile the plane using any kind of reflection about one of its sides, and since an isosceles right triangle of any size can easily cover a square, we can conclude that these triangles could have extendable solutions.

However, the ability to tile the entire plane in this manner does not guarantee completeness. Reflecting over a side of the triangle under homogeneous conditions creates an anti-symmetric re-orientation of the solution. For the triangle T and any point $t \in T$, if $f(t)$ is the value of the solution function at t and t' is the reflection of t about a side of T, then $f(t') = -f(t)$. However, if different reflections overlap themselves in such a way that a point with positive value lies on a point with a negative value that is equal in magnitude, they will zero each other out, making the extended solution trivial.

To illustrate this, we consider the 30-30-120 triangle, which we know to tile the plane because it is the combination of two 30-60-90 triangles joined along their shortest edge. Solutions which are anti-symmetric are simply an extension of these 30-60-90 triangles with homogeneous conditions; however, symmetric solutions are needed to obtain a complete solution set so that we can have functions that do not need to be zero along the altitude. Reflections of a symmetric solution appear in Figure 3. The bold lines outline the original triangle, and the "+" and "−" signs illustrate the symmetry about the dashed line within each triangle and the anti-symmetry of reflecting about an edge of the triangle. The circled minus signs show why a symmetric solution cannot exist in the 30-30-120 triangle. Anti-symmetry must exist about solid lines, as in a plus and a minus on each side, but these minus signs violate that condition, making symmetric solutions inconsistent and a complete solution set impossible. Through this analysis, we actually were able to determine which triangles could have complete solution sets: the equilateral triangle, the 30-60-90 triangle, and the isosceles right triangle.

McCartin actually addresses in [7] the concept of which domains in the $\mathbb{R}^2$ plane can have complete trigonometric solutions. He presents geometric arguments showing that the only polygonal domains capable of having complete trigonometric solutions with either homogeneous Dirichlet or Neumann conditions are the square, the rectangle, and the three triangles mentioned above. The rectangle and square have standardized solution sets that can be found in any partial differential equation textbook, and the equilateral triangle and 30-60-90 triangle have solution sets given in [5] and [4]. However, McCartin does not present solutions for the isosceles right triangle, and as indicated above, our application of McCartin's method did not prove to be applicable to non-equilateral triangles. Instead another method developed by Milan Prager proved to be extendable to isosceles right triangles, and we present this method in the next section.

4 Prager's Method: Adaptation to Isosceles Right Triangles

Prager in [9] formulates solutions to Laplace's equation under homogeneous Dirichlet conditions for the equilateral triangle and the 30-60-90 triangle by "folding" a rectangle with sides of length 1 and $\sqrt{3}$ into the appropriate triangle. However, the folding can be reversed using reflections, so solution sets in either the triangle or the rectangle can be transformed into a solution set in the other two-dimensional region.

Since an isosceles right triangle can be reflected over it's hypotenuse to form a square, we infer that Prager's method could be easily applied to find a complete solution set for the isosceles right triangle. What follows is the results of applying Prager's method to the isosceles right triangle.

4.1 Coordinate System and Function Transformations

We begin our application of Prager's method by defining our geometry in $\mathbb{R}^2$, which appears in Figure 4. While McCartin's method used symmetry about one of the triangle's altitudes, Prager's method simply uses one where the triangle is embedded into a rectangle through reflections about a side of the triangle. Our ultimate goal is to determine

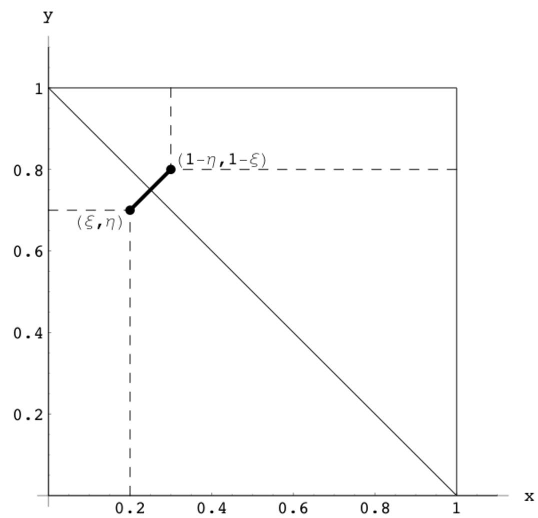


Figure 4: Coordinate System for Prager's Method

a set of functions $u \in L_2(T_1)$ that solve our equation and boundary conditions, where T_1 is the isosceles right triangle with vertices at $(0,0)$, $(1,0)$, and $(0,1)$. We then use a transformation known as a prolongation, $\mathfrak{P}$, of the function from points in T_1 to points in the square $S = (0,1) \times (0,1)$. This transformation is obtained by reflecting T_1 over its hypotenuse to create anti-symmetry in T_2. Thus, we can define the corresponding points by:

$$\begin{aligned} x_1 &= \xi, & y_1 &= \eta, \\ x_2 &= 1-\eta, & y_2 &= 1-\xi. \end{aligned}$$

Prager also uses the notational shorthand $B_i = (x_i, y_i) \in T_i$ for points that correspond to each other through reflection, as well as saying that $B = (\xi, \eta) \in T_1$. For a more comprehensive description of this notation, see [9]. Also, the prolongation of the function $u \in L_2(T_1)$ is defined as:

$$\mathfrak{P}u(B_i) = c_i u(B_i), \text{ on } T_i \tag{47}$$

where $c_1 = 1$ and $c_2 = -1$.

Just as we can extend a function on T_1 onto S, we can also transform a function $v \in L_2(S)$ onto T_1 using a folding transformation, $\mathfrak{F}$. This folding transformation is essentially the opposite of the prolongation transformation, since it takes the square S and folds it across the hypotenuse of T_1. The expression for this transformation is as follows:

$$\mathfrak{F}v(B) = \sum_{i=1}^{2} c_i v(B_i). \tag{48}$$

Note that in (48), we are evaluating the value of $\mathfrak{F}[v]$ at points $B \in T_1$.

4.2 Significance of the Transformations

Of the two transformations presented in (47) and (48), the folding transformation appears to be the most useful. Why? First, let the function $v \in L_2(S)$ satisfy Helmholtz's equation on the square with homogeneous Dirichlet boundary conditions, as in,

$$\nabla^2 v = \lambda v, \tag{49}$$

where $v = 0$ on the edges of the square. We now apply the folding transformation and see what we can conclude about $\mathfrak{F}v \in L_2(T_1)$:

$$\begin{aligned}
\nabla^2(\mathfrak{F}v) &= \nabla^2\left(\sum_{i=1}^{2} c_i v(B_i)\right) = \sum_{i=1}^{2} \nabla^2\left(c_i v(B_i),\right) \\
&= \sum_{i=1}^{2} c_i \nabla^2 v(B_i) = \sum_{i=1}^{2} c_i \lambda \cdot v(B_i), \\
&= \lambda(v(B_1) - v(B_2)) = 2\lambda v(B).
\end{aligned} \tag{50}$$

Thus, $\mathfrak{F}v$ is also an eigenfunction of the Laplace operator. Notice that in (50), we can substitute $v(B_2) = -v(B)$ because of the anti-symmetric reflection.

If $v = 0$ along the edges of the square, then $\mathfrak{F}v = 0$ along the legs of the triangle since the folding transformation would only be adding up zeros. Along the hypotenuse of the triangle, we know that $v(B_1)$ and $v(B_2)$ are both equal to $v(B)$, so the folding transformation of the function evaluates to:

$$\mathfrak{F}v(B) = \sum_{i=1}^{2} c_i v(B_i) = v(B_1) - v(B_2) = v(B) - v(B) = 0. \tag{51}$$

Hence, we can conclude that $\mathfrak{F}v$ is zero along all three edges of the isosceles right triangle, so it satisfies the homogeneous boundary conditions. Furthermore, because these transformations are essentially equivalent to ones used by Prager we can conclude that $\mathfrak{F}v$ must satisfy Laplace's equation for the same reasons that Prager presents in [9].

4.3 Base Functions for the Fourier Series

In order to find the base functions for the Fourier series on the isosceles right triangle, we can take the base functions for the square and simply apply the folding transformation to them. The homogeneous boundary conditions for the square can be described by the following expression:

$$v(x = 0, y) = v(x = 1, y) = v(x, y = 0) = v(x, y = 1) = 0. \tag{52}$$

The canonical eigenfunctions are known to be:

$$v_{k.l} = \sin k\pi x \sin l\pi y, \tag{53}$$

where $k = 1, 2, 3, \ldots$ and $l = 1, 2, 3, \ldots$ form the domains for the eigenvalues, $k\pi$ and $l\pi$ [2]. We know this expression satisfy the boundary conditions because $\sin 0 = 0$ and for any integer m, $\sin m\pi = 0$.

After we apply the transformation in (48) and use the trigonometric identity

$$\sin(a-b) = \sin a \cos b - \cos a \sin b, \tag{54}$$

we get the following expression for the base function of the isosceles right triangle:

$$\begin{aligned}\mathfrak{F}v_{k,l} &= \sin k\pi x \sin l\pi y - \sin k\pi(1-y)\sin l\pi(1-x), \\ &= \sin k\pi x \sin l\pi y - (\sin k\pi \cos k\pi y - \cos k\pi \sin k\pi y) \\ &\quad \cdot (\sin l\pi \cos l\pi x - \cos l\pi \sin l\pi x)\,. \end{aligned} \tag{55}$$

Because k and l are integers, we can say

$$\begin{aligned} \sin k\pi = 0, &\qquad \sin l\pi = 0, \\ \cos k\pi = (-1)^k, &\qquad \cos l\pi = (-1)^l. \end{aligned} \tag{56}$$

Equation (55) becomes:

$$\begin{aligned}\mathfrak{F}v_{k,l} &= \sin k\pi x \sin l\pi y - \left(0 - (-1)^k \sin k\pi y\right)\left(0 - (-1)^l \sin l\pi x\right), \\ &= \sin k\pi x \sin l\pi y + (-1)^{k+l+1} \sin k\pi y \sin l\pi x. \end{aligned} \tag{57}$$

We now have an expression for the eigenfunctions for the isosceles right triangle, which means we have discovered the basic eigenstructure of this triangle!

4.4 Plots of Eigenfunctions

Using the expression in Equation (57), we can actually plot the eigenfunctions for certain values of k and l and illustrate their features.

Before we actually produce plots, we mention that some functions in (57) would be trivial despite being in the theoretical domain for values of k and l. First of all, switching the values of k and l would not yield significantly different plots because of the symmetry of the triangle, as in, the plot for $(k,l) = (2,3)$ would just be the plot for $(k,l) = (3,2)$ reflected about the line $y = x$.

Secondly, when $k = l$, Equation (57) becomes a complicated expression for zero, so we omit such plots. This is actually a more interesting result then one thinks. Typically, when $k = l$, the square is divided into smaller squares, but because of the inherent symmetry of this action, the folding transformation actually annihilates the entire function. As a result, we cannot have a $(1,1)$-mode, but as we will show, the unimodal structure of this basic mode actually appears in the $(1,2)$-mode.

The first two plots we present are perhaps the most simple, non-trivial functions available: $(k,l) = (1,2)$ and $(k,l) = (1,3)$. These plots appear in Figures 5 and 6.

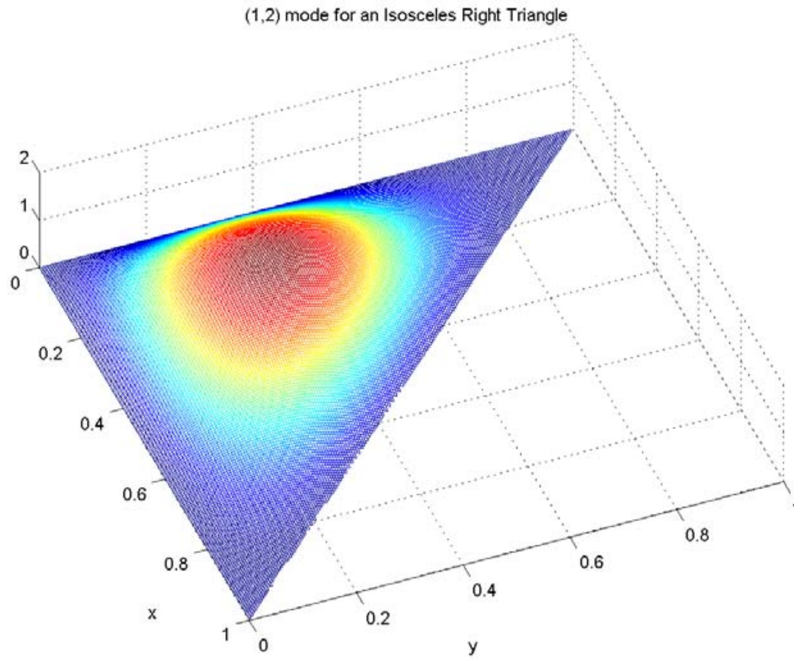


Figure 5: (1, 2)-Mode for an Isosceles Right Triangle

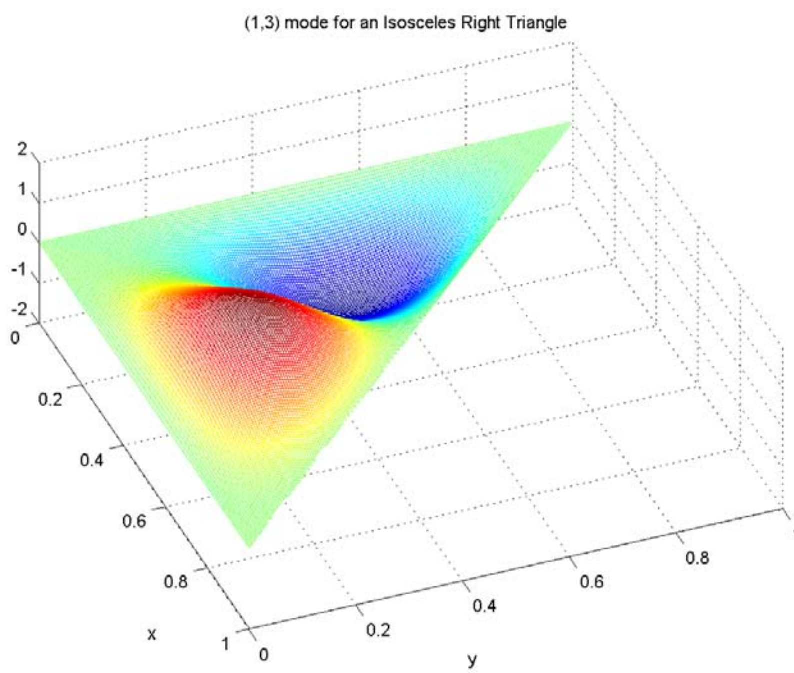


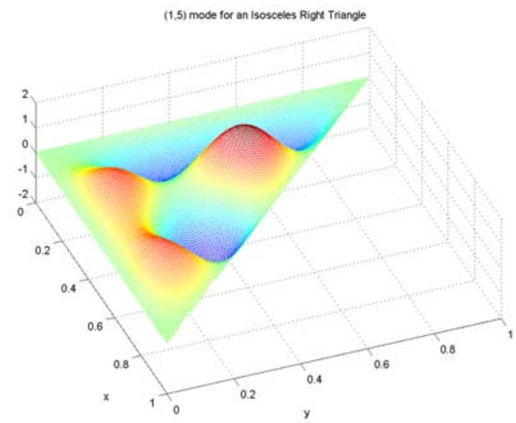
Figure 6: (1, 3)-Mode for an Isosceles Right Triangle

In both of these figures, the functions clearly satisfy the homogeneous Dirichlet boundary conditions. Furthermore, the (1, 2) mode is a symmetric mode, and the (1, 3) mode is anti-symmetric. This means that any function $u \in L_2(T_1)$ can be decomposed into the sum of symmetric and anti-symmetric modes. Additional images can be found in the appendix.

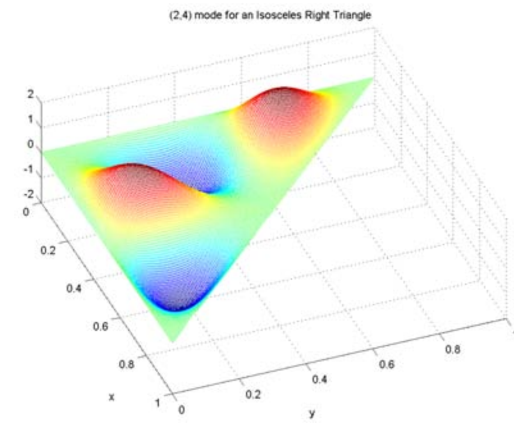
5 Conclusion

In this paper we extended a solution method for the eigenvalue problem in a equilateral triangle to general isosceles triangles. However, we did not find solutions, instead we managed to show that the solution method in [5] did not extend to general isosceles triangles. We showed that the problem could be solved in a right isosceles triangle and investigated a method of folding rectangles to develop solutions. Coupling Prager's method with our work on general isosceles triangles and through the use of a geometric argument, we were able to show that the only isosceles triangles for which there are complete solutions are the equilateral triangle and the right isosceles triangle.

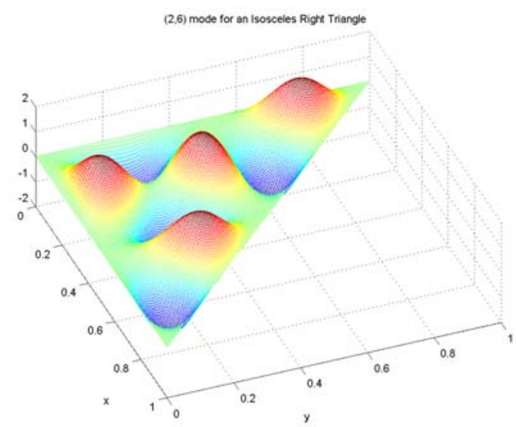
A Additional Images



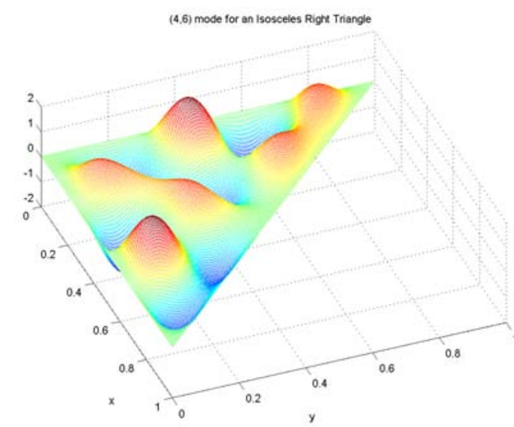
(a) $(1, 5)$-Mode for an Isosceles Right Triangle



(b) $(2, 4)$-Mode for an Isosceles Right Triangle



(c) $(2, 6)$-Mode for an Isosceles Right Triangle



(d) $(4, 6)$-Mode for an Isosceles Right Triangle

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