

Analytical Solution to the Large Deformation of Cantilever Beams with Two Angled Point Forces

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Abstract

The analytical solution for the large deformation of a cantilever beam under a point load, typically applied to the tip of the cantilever and perpendicularly to its axis, has been widely studied and published. However, the more complex case of two angled point loads applied to the cantilever has not been published. The current research delved into the following scenario: an upright cantilever, e.g. a pole, has point loads applied at two locations on the cantilever, where each point load is angled, i.e. the point load has both a horizontal component (which may result from wind loading) and downward vertical component of force (such as from weights). The aim of the research is to develop a methodology for finding, at the two locations where the point loads are applied, the angle of deflection, the horizontal deflection, and the vertically deflected height. Ultimately, the research yielded a methodology based on the Complete and Incomplete Elliptic Integrals of the First Kind and Second Kind. The analytical solution developed in this research - specifically the method for calculating the angles of deflections - was compared against Finite Element Analysis and was found to produce nearly identical results. We conclude that the methodology shown can be extended to any number of point loads and will be a contribution to the field of non-linear mechanics.

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1. Introduction

The analytical solution to the large deformation of a cantilever beam under a tip load, i.e. a point load or force applied to the tip of the cantilever, has been studied thoroughly, as shown in [1] - [7]. Another popular topic of study is the large deformation of a cantilever under a tip load and a distributed load, as shown in [8] - [12]. Indeed, even a master's thesis has been written on the subject of a cantilever beam under a single tip load, as in [13]. The current research examines a novel and more complex topic: the large deformation of a cantilever due to two angled point loads. Specifically, the research proposes this problem: an upright cantilever, e.g. a pole, has two angled point loads, meaning each point load has both a horizontal and vertical component of force. See Figure 3, right image. The motivation for the study is that solving the problem of two angled point loads (where the angles do not have to be the same) provides a roadmap for solving the problem of any number of angled point loads along the cantilever. The result of this work is that the analytical solution for the proposed problem has been found: the work provides methodology that can be used to calculate the cantilever horizontal deflection, vertical deflection, and tip angle at each angled point load. This work is of interest to persons involved in the design and construction of utility poles, transmission towers, cell towers, antennas, and any other pole-like structure that has multiple horizontal loads (such as wind blowing on exposed surfaces) and vertical loads (such as the weight of the structure or its attachments).

Here, we will briefly discuss two important developmental steps in non-linear mechanics to see how the current work is novel. Before we do so, we will observe the following two definitions: (1) non-angled: this means that, prior to cantilever deflection, the force is exactly perpendicular to the length of the cantilever; (2) angled: this means that, prior to cantilever deflection, the force is at some arbitrary angle to the length of the cantilever. One of the earliest works exploring the deformation of a cantilever is [14] from Bisshopp and Drucker. In this paper, the authors describe a solution for the deflection of a cantilever under one non-angled point load at the tip of the cantilever. Another well-known work is [15] from Frisch-Fay. The author describes a solution for the deflection of a cantilever under two or more angled point loads as long as all of the loads are parallel to each other. By using parallel loads, the author was able to provide a simplified approach utilizing the principle of elastic similarity. In the current work, we provide essentially the general solution for solving the deflection of a cantilever under an arbitrary loading condition. This general solution is not provided in [14] or [15]. The solution in the current work can be extended to any number of loads, with each load able to have an arbitrary and different angle.

2. Large Deformation using One Prismatic Element

We first consider a pole with a tip load, where the tip load is comprised of both a horizontal force P_T and a vertical force P_V . The pole is treated as a single prismatic element of length L . This scenario is shown in Figure 1 below. We will determine, at the pole tip, the angle of deflection ϕ_o , the horizontal deflection δ_x , and the vertically deflected height $L - \delta_y$.

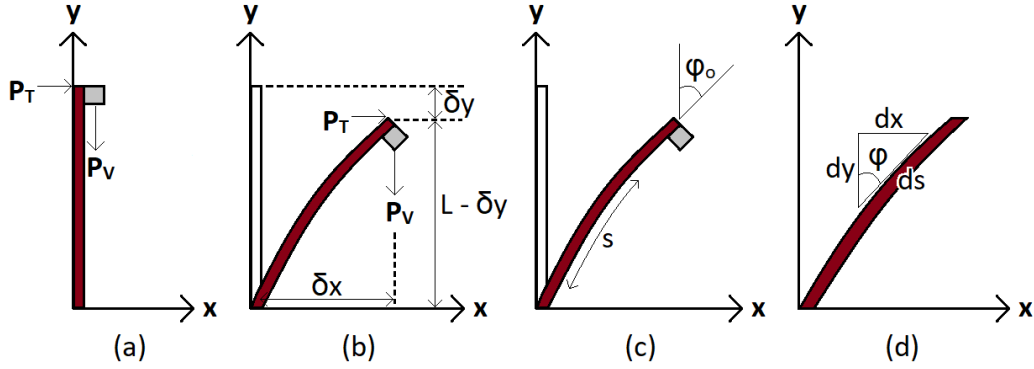


Figure 1: (a) Prismatic pole with a horizontal (or *transverse*) load and a vertical load, prior to deflection. (b) The pole after deflection. (c) The arc length s and the angle of deflection ϕ . Note that the angle of deflection at the pole tip is denoted as ϕ_o . (d) Here, we show a small, approximately straight-line segment.

2.1. Angle of Deflection at the Element Tip

As a first step toward determining a pole's vertical and horizontal deflection, let us calculate ϕ_o . We define E as the pole's elasticity and I as the pole's moment of inertia (or second moment of area). Bernoulli-Euler beam theory gives the following relationship between a beam's curvature, $d\phi/ds$, and its bending moment, M .

$$M = EI \cdot \frac{d\phi}{ds} \quad (1)$$

To work with differentials, let us differentiate both sides of (1) with respect to s , as follows:

$$\frac{dM}{ds} = EI \cdot \frac{d^2\phi}{ds^2} \quad (2)$$

The bending moment at any location (x,y) along the pole is:

$$M(s) = P_T \cdot (L - \delta_y - y) + P_V \cdot (\delta_x - x) \quad (3)$$

Differentiating both sides of (3) gives:

$$\frac{dM}{ds} = P_T \cdot \left(\frac{d}{ds}L - \frac{d}{ds}\delta_y - \frac{dy}{ds} \right) + P_V \cdot \left(\frac{d}{ds}\delta_x - \frac{dx}{ds} \right) \quad (4)$$

By examining Figure 1(d) we can obtain the following two relationships:

$$\frac{dx}{ds} = \sin \phi \quad (5)$$

$$\frac{dy}{ds} = \cos \phi \quad (6)$$

Substituting (5) and (6) into (4), we obtain:

$$\frac{dM}{ds} = -P_T \cdot \cos \phi - P_V \cdot \sin \phi \quad (7)$$

Substituting (7) into (2), we obtain:

$$EI \cdot \frac{d^2\phi}{ds^2} + P_T \cdot \cos \phi + P_V \cdot \sin \phi = 0 \quad (8)$$

Multiply both sides of (8) by $d\phi/ds$ to obtain:

$$EI \cdot \frac{d^2\phi}{ds^2} \cdot \frac{d\phi}{ds} + P_T \cdot \cos \phi \cdot \frac{d\phi}{ds} + P_V \cdot \sin \phi \cdot \frac{d\phi}{ds} = 0 \quad (9)$$

We can rewrite this as follows:

$$\frac{d}{ds} \left[\frac{1}{2} \cdot EI \cdot \left(\frac{d\phi}{ds} \right)^2 + P_T \cdot \sin \phi - P_V \cdot \cos \phi \right] = 0 \quad (10)$$

From which we obtain:

$$\frac{1}{2} \cdot EI \cdot \left(\frac{d\phi}{ds} \right)^2 + P_T \cdot \sin \phi - P_V \cdot \cos \phi = C \quad (11)$$

In order to determine the constant C , we use the condition that when $(\phi, s) = (\phi_o, L)$, it follows that $d\phi/ds = 0$. This indicates that at the tip of the beam, the curvature is 0, i.e. the radius of curvature is infinity because if we were to extend the tip of the beam, that extension would continue in a straight line. Applying our condition, we can find our constant as:

$$C = P_T \cdot \sin \phi_o - P_V \cdot \cos \phi_o \quad (12)$$

We substitute (12) into (11) and solve for ds :

$$ds = \sqrt{\frac{EI}{2}} \cdot \frac{d\phi}{\sqrt{P_T \cdot \sin \phi_o - P_T \cdot \sin \phi - P_V \cdot \cos \phi_o + P_V \cdot \cos \phi}} \quad (13)$$

By performing a summation of sinusoids in the denominator of the second factor, (13) can be simplified as follows:

$$ds = \sqrt{\frac{EI}{2}} \cdot \frac{d\phi}{\sqrt{P_R \cdot \sin(\phi_o + \beta) - P_R \cdot \sin(\phi + \beta)}} \quad (14)$$

where

$$P_R = \sqrt{P_T^2 + P_V^2} \quad (15)$$

$$\beta = \arcsin \left(\frac{-P_V}{\sqrt{P_T^2 + P_V^2}} \right) \quad (16)$$

We can integrate both sides of (14) along the entire length of the pole, starting from the base where $\phi = 0$ to the pole tip, where $\phi = \phi_o$.

$$L = \sqrt{\frac{EI}{2}} \cdot \int_0^{\phi_o} \frac{d\phi}{\sqrt{P_R \cdot \sin(\phi_o + \beta) - P_R \cdot \sin(\phi + \beta)}} \quad (17)$$

If we further define a value α as:

$$\alpha = \frac{P_R \cdot L^2}{EI} \quad (18)$$

then (17) can be rewritten as:

$$\sqrt{\alpha} = \int_0^{\phi_o} \frac{1}{\sqrt{2}} \cdot \frac{d\phi}{\sqrt{\sin(\phi_o + \beta) - \sin(\phi + \beta)}} \quad (19)$$

As it stands, (19) is not integrable and has no known function approximation. However, it can be simplified into an elliptic integral which *does* have a function approximation. The following equations will show, step by step, how to perform the needed manipulations. First, in (19), we multiply the numerator and denominator of the integrand by a cosine value:

$$\sqrt{\alpha} = \int_0^{\phi_o} \frac{1}{\sqrt{2}} \cdot \frac{\cos(\phi + \beta) \cdot d\phi}{\sqrt{1 - \sin^2(\phi + \beta)} \cdot \sqrt{\sin(\phi_o + \beta) - \sin(\phi + \beta)}} \quad (20)$$

The first term in the denominator of (20) can be factorized, and then the integrand rearranged, as follows:

$$\begin{aligned} \sqrt{\alpha} = & \int_0^{\phi_o} \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{1 + \sin(\phi + \beta)} \cdot \sqrt{\sin(\phi_o + \beta) - \sin(\phi + \beta)}} \\ & \cdot \frac{\cos(\phi + \beta) \cdot d\phi}{\sqrt{1 - \sin(\phi + \beta)}} \end{aligned} \quad (21)$$

In (21), we multiply both the numerator and denominator of the integrand by the same sine value:

$$\begin{aligned} \sqrt{\alpha} = & \int_0^{\phi_o} \frac{1}{\sqrt{2}} \cdot \frac{1 + \sin(\phi_o + \beta)}{\sqrt{1 + \sin(\phi + \beta)} \cdot \sqrt{\sin(\phi_o + \beta) - \sin(\phi + \beta)}} \\ & \cdot \frac{\cos(\phi + \beta) \cdot d\phi}{[1 + \sin(\phi_o + \beta)] \cdot \sqrt{1 - \sin(\phi + \beta)}} \end{aligned} \quad (22)$$

Now here is where we do a substitution. Let the following be true:

$$\sin \psi = \sqrt{\frac{1 + \sin(\phi + \beta)}{1 + \sin(\phi_o + \beta)}} \quad (23)$$

By using a sine identity, (23) also yields:

$$\cos \psi = \sqrt{\frac{\sin(\phi_o + \beta) - \sin(\phi + \beta)}{1 + \sin(\phi_o + \beta)}} \quad (24)$$

Using both (23) and (24), we can also write:

$$\sin 2\psi = 2 \cdot \frac{\sqrt{1 + \sin(\phi + \beta)} \cdot \sqrt{\sin(\phi_o + \beta) - \sin(\phi + \beta)}}{1 + \sin(\phi_o + \beta)} \quad (25)$$

Squaring both sides of (23) gives:

$$\sin^2 \psi = \frac{1 + \sin(\phi + \beta)}{1 + \sin(\phi_o + \beta)} \quad (26)$$

Taking the derivative of both sides of (26) gives:

$$\sin 2\psi d\psi = \frac{\cos(\phi + \beta)}{1 + \sin(\phi_o + \beta)} \cdot d\phi \quad (27)$$

Substituting (25) and (27) into (22) and simplifying, we get:

$$\begin{aligned} \sqrt{\alpha} &= \int_0^{\phi_o} \frac{\sqrt{2} \cdot d\psi}{\sqrt{1 - \sin(\phi + \beta)}} \\ &= \int_0^{\phi_o} \frac{d\psi}{\sqrt{1 - \frac{1 + \sin(\phi + \beta)}{2}}} \end{aligned} \quad (28)$$

Note that we have not yet changed the limits of integration. We will wait until we make a second substitution. At this point, we invent a value k such that the following be true:

$$k^2 \cdot \sin^2 \psi = \frac{1 + \sin(\phi + \beta)}{2} \quad (29)$$

Substitute (29) into (28) to obtain:

$$\sqrt{\alpha} = \int_{\phi=0}^{\phi=\phi_o} \frac{d\psi}{\sqrt{1 - k^2 \cdot \sin^2 \psi}} \quad (30)$$

Now we can change the limits of integration. Substituting $\phi = \phi_o$ into (26) and solving for ψ , the upper limit of integration becomes:

$$\begin{aligned} \psi(\phi = \phi_o) &= \arcsin \sqrt{\frac{1 + \sin(\phi_o + \beta)}{1 + \sin(\phi_o + \beta)}} \\ &= \frac{\pi}{2} \end{aligned} \quad (31)$$

Substituting $\phi = 0$ into (29) and solving for ψ , the lower limit of integration becomes:

$$\psi(\phi = 0) = \psi_o = \arcsin \left(\frac{1}{\sqrt{2} \cdot k} \cdot \sqrt{1 + \sin \beta} \right) \quad (32)$$

We can rewrite (30) as:

$$\sqrt{\alpha} = \int_{\psi_o}^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1 - k^2 \cdot \sin^2 \psi}} \quad (33)$$

Here let us make a quick note about k - it is some constant whose value is unknown (we will explain how to deal with this later). However, we are able to determine a range of possible values for k . By substituting (26) into (29) and solving for k , we find:

$$k = \frac{1}{\sqrt{2}} \cdot \sqrt{1 + \sin(\phi_o + \beta)} \quad (34)$$

Since we have $0 \leq \sin(\phi_o + \beta) \leq 1$ we deduce that $\sqrt{2}/2 \leq k \leq 1$. Now, we observe that (33) can be rewritten as follows:

$$\sqrt{\alpha} = \int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1 - k^2 \cdot \sin^2}} - \int_0^{\psi_o} \frac{d\psi}{\sqrt{1 - k^2 \cdot \sin^2}} \quad (35)$$

We note that in (35) the first term is the Complete Elliptic Integral of the First Kind, and the second term is the Incomplete Elliptic Integral of the First Kind. With this in mind, we can rewrite (35) as:

$$\sqrt{\alpha} = F_n\left(\frac{\pi}{2}, k, 0\right) - F_n(\psi_o, k, 0) \quad (36)$$

where:

$$F_n\left(\frac{\pi}{2}, k, 0\right) = \frac{\pi}{2(2n+1)} \cdot \left[1 + 2 \cdot \sum_{m=1}^n \left(1 - \left[k \cdot \sin\left(\frac{m\pi}{2n+1}\right) \right]^2 \right)^{-1/2} \right] \quad (37)$$

$$F_n(\psi_o, k, 0) = \frac{1}{2n+1} \cdot \left[\psi_o + 2 \cdot \sum_{m=1}^n \left(1 - \left[k \cdot \sin\left(\frac{m\pi}{2n+1}\right) \right]^2 \right)^{-1/2} \right. \\ \left. \cdot \arctan \left\{ \tan \psi_o \cdot \left(1 - \left[k \cdot \sin\left(\frac{m\pi}{2n+1}\right) \right]^2 \right)^{1/2} \right\} \right] \quad (38)$$

The variable n is the number of terms that you want to expand each elliptic integral. Typically, $n = 10$ or even less will yield sufficient accuracy. Recall that our goal was to find ϕ_o . The process is as follows: 1) Guess k until (36) is true, 2) Solve (34) for ϕ_o .

2.2. Horizontal Deflection at the Pole Tip

After finding ϕ_o we can determine the horizontal deflection δ_x at the pole tip. To find δ_x , we substitute (14) into (5), solve for dx , and then integrate both sides from $\phi = 0$ to $\phi = \phi_o$ to obtain:

$$\delta_x = \sqrt{\frac{EI}{2}} \cdot \int_0^{\phi_o} \frac{\sin \phi \cdot d\phi}{\sqrt{P_R \cdot \sin(\phi_o + \beta) - P_R \cdot \sin(\phi + \beta)}} \quad (39)$$

As it stands, (39) is not integrable and only numerical methods can be used. However, elliptic integrals will once again provide us with a simplified alternative. We define a value α_x as:

$$\alpha_x = \frac{P_R \cdot \delta_x^2}{EI} \quad (40)$$

Substituting (40) into (39) will yield:

$$\sqrt{\alpha_x} = \int_0^{\phi_o} \frac{1}{\sqrt{2}} \cdot \frac{\sin \phi \cdot d\phi}{\sqrt{\sin(\phi_o + \beta) - \sin(\phi + \beta)}} \quad (41)$$

By using the same substitutions as in (23) and (29), we can arrive at an equation that is analogous to (33):

$$\sqrt{\alpha_x} = \int_{\psi_o}^{\frac{\pi}{2}} \frac{\sin \phi \cdot d\psi}{\sqrt{1 - k^2 \cdot \sin^2}} \quad (42)$$

It is obvious that we must find a way to represent the $\sin \phi$ term in the numerator as some function of ψ . To this end, we can take (29) and solve for ϕ :

$$\phi = \arcsin(2k^2 \cdot \sin^2 \psi - 1) - \beta \quad (43)$$

If we take the sine of both sides of (43) and expand, we get:

$$\sin \phi = (2k^2 \cdot \sin^2 \psi - 1) \cdot \cos \beta - \sin \beta \cdot 2k \cdot \sin \psi \cdot \sqrt{1 - k^2 \cdot \sin^2} \quad (44)$$

Substitute (44) into (42):

$$\begin{aligned} \sqrt{\alpha_x} &= \cos \beta \cdot \int_{\psi_o}^{\frac{\pi}{2}} \frac{2k^2 \sin^2 \psi - 1}{\sqrt{1 - k^2 \sin^2}} \cdot d\psi - 2k \sin \beta \cdot \int_{\psi_o}^{\frac{\pi}{2}} \sin \psi d\psi \\ &= \cos \beta \cdot \int_{\psi_o}^{\frac{\pi}{2}} \frac{-2(1 - k^2 \sin^2 \psi) + 1}{\sqrt{1 - k^2 \sin^2}} \cdot d\psi + 2k \sin \beta \cdot \cos \psi \Big|_{\psi_o}^{\pi/2} \\ &= -2 \cos \beta \cdot \int_{\psi_o}^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2} \psi \cdot d\psi + \cos \beta \cdot \int_{\psi_o}^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2}} \cdot d\psi \\ &\quad + 2k \sin \beta \cdot (0 - \cos \psi_o) \end{aligned} \quad (45)$$

As we did with (35), we can rewrite the first term of (45) as the difference of the Complete and Incomplete Elliptic Integrals of the Second Kind, and the second term as the difference of the Complete and Incomplete Elliptic Integral of the First Kind:

$$\begin{aligned} \sqrt{\alpha_x} &= -2 \cos \beta \cdot \left[E_n \left(\frac{\pi}{2}, k \right) - E_n (\psi_o, k) \right] + \cos \beta \cdot \left[F_n \left(\frac{\pi}{2}, k, 0 \right) - F_n (\psi_o, k, 0) \right] \\ &\quad - 2k \sin \beta \cos \psi_o \end{aligned} \quad (46)$$

where:

$$E_n \left(\frac{\pi}{2}, k \right) = \frac{\pi}{2} \cdot \left[(2n + 1) - 2 \cdot (2n + 1)^{-1} \cdot \sum_{m=1}^n \frac{\tan^2 \theta_m}{\rho_m} \right] \quad (47)$$

$$E_n (\psi_o, k) = (2n + 1) \cdot \psi_o - 2 \cdot (2n + 1)^{-1} \cdot \sum_{m=1}^n \frac{\tan^2 \theta_m \cdot \arctan (\rho_m \cdot \tan \psi_o)}{\rho_m} \quad (48)$$

$$\theta_m = \frac{m\pi}{2n + 1} \quad (49)$$

$$\rho_m = \sqrt{1 - k^2 \cdot \cos^2 \theta_m} \quad (50)$$

From our prior efforts in solving for ϕ_o we would have already found β , k , and ψ_o , as well as the difference of the Complete and Incomplete Elliptic Integrals of the First Kind. After solving (46) for α_x , we can substitute this value into (40) to find δ_x .

2.3. Vertically Deflected Height at the Pole Tip

Similarly, to find $L - \delta_y$ (and δ_y), substitute (14) into (6), solve for dy , and then integrate both sides from $\phi = 0$ to $\phi = \phi_o$ to obtain:

$$L - \delta_y = \sqrt{\frac{EI}{2}} \cdot \int_0^{\phi_o} \frac{\cos \phi \cdot d\phi}{\sqrt{P_R \cdot \sin(\phi_o + \beta) - P_R \cdot \sin(\phi + \beta)}} \quad (51)$$

As it stands, (51) is not integrable. However, elliptic integrals can be used to obtain a simplified alternative. We define a value α_y as:

$$\alpha_y = \frac{P_R \cdot (L - \delta_y)^2}{EI} \quad (52)$$

Substituting (52) into (51) will yield:

$$\sqrt{\alpha_y} = \int_0^{\phi_o} \frac{1}{\sqrt{2}} \cdot \frac{\cos \phi \cdot d\phi}{\sqrt{\sin(\phi_o + \beta) - \sin(\phi + \beta)}} \quad (53)$$

By using the same substitutions as in (23) and (29), we can arrive at an equation that is analogous to (33):

$$\sqrt{\alpha_y} = \int_{\psi_o}^{\frac{\pi}{2}} \frac{\cos \phi \cdot d\psi}{\sqrt{1 - k^2 \cdot \sin^2 \psi}} \quad (54)$$

It is obvious that we must find a way to represent the $\cos \phi$ term in the numerator as some function of ψ . To this end, we can take the cosine of both sides of (43), expand, and obtain:

$$\cos \phi = (2k^2 \cdot \sin^2 \psi - 1) \cdot \sin \beta + 2k \cdot \sin \psi \cdot \sqrt{1 - k^2 \sin^2 \psi} \cdot \cos \beta \quad (55)$$

Substitute (55) into (54):

$$\begin{aligned} \sqrt{\alpha_y} &= \sin \beta \cdot \int_{\psi_o}^{\frac{\pi}{2}} \frac{2k^2 \sin^2 \psi - 1}{\sqrt{1 - k^2 \sin^2 \psi}} \cdot d\psi + 2k \cos \beta \cdot \int_{\psi_o}^{\frac{\pi}{2}} \sin \psi d\psi \\ &= \sin \beta \cdot \int_{\psi_o}^{\frac{\pi}{2}} \frac{-2(1 - k^2 \sin^2 \psi) + 1}{\sqrt{1 - k^2 \sin^2 \psi}} \cdot d\psi - 2k \cos \beta \cdot \cos \psi \Big|_{\psi_o}^{\pi/2} \\ &= -2 \sin \beta \cdot \int_{\psi_o}^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \psi} \cdot d\psi + \sin \beta \cdot \int_{\psi_o}^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2 \psi}} \cdot d\psi \\ &\quad - 2k \cos \beta \cdot (0 - \cos \psi_o) \end{aligned} \quad (56)$$

As we did with (35), we can rewrite the first term of (56) as the difference of the Complete and Incomplete Elliptic Integrals of the Second Kind, and the second term as the difference of the Complete and Incomplete Elliptic Integral of the First Kind:

$$\begin{aligned} \sqrt{\alpha_y} &= -2 \sin \beta \cdot \left[E_n \left(\frac{\pi}{2}, k \right) - E_n (\psi_o, k) \right] + \sin \beta \cdot \left[F_n \left(\frac{\pi}{2}, k, 0 \right) - F_n (\psi_o, k, 0) \right] \\ &\quad + 2k \cos \beta \cos \psi_o \end{aligned} \quad (57)$$

where every variable and term is already known from our previous efforts at finding ψ_o and the horizontal deflection. After solving (57) for α_y , we can substitute this value into (52) to find $L - \delta_y$, from which we can also obtain the vertical deflection δ_y .

3. Large Deformation using Two Prismatic Elements

We now consider a pole with point loads at two locations, as shown on the right side of Figure 3: at some Point 1, a horizontal force $P_{1,T}$ and a vertical force $P_{1,V}$ is applied; at some Point 2, a horizontal force $P_{2,T}$ and a vertical force $P_{2,V}$ is applied. The pole is treated as two prismatic elements: the length L_1 between the base and Point 1 is considered the first element (“Segment 1”); the length L_2 between Point 1 and Point 2 is considered the second element (“Segment 2”). At Point 1, we will determine the angle of deflection ϕ_1 , the horizontal deflection δ_{x1} , and the vertically deflected height a ; at Point 2, we will determine the angle of deflection ϕ_2 , the horizontal deflection δ_{x2} (which is with deflection with respect to Point 1), and the vertically deflected height b (which results from the downward deflection of the second element only and is independent of any bending in the first element).

3.1. Angles of Deflection at the Element Tips

We first focus on Segment 2, which appears as a pole with one load. Therefore, we can apply (26) and (29) to Segment 2 as follows:

$$\sin^2 \psi = \frac{1 + \sin(\phi + \beta_2)}{1 + \sin(\phi_2 + \beta_2)} \quad (58)$$

$$k_2^2 \cdot \sin^2 \psi = \frac{1 + \sin(\phi + \beta_2)}{2} \quad (59)$$

where, like (16), we can express β_2 as:

$$\beta_2 = \arcsin \left(\frac{-P_{2,V}}{\sqrt{P_{2,T}^2 + P_{2,V}^2}} \right) \quad (60)$$

By substituting (59) into (58) and solving, we get:

$$\sin(\phi_2 + \beta_2) = 2 \cdot k_2^2 - 1 \quad (61)$$

Finally:

$$L_2 = \sqrt{\frac{EI_2}{P_{2,R}}} \cdot \left[F_n \left(\frac{\pi}{2}, k_2, 0 \right) - F_n \left(\psi_2|_{\phi=\phi_1}, k_2, 0 \right) \right] \quad (62)$$

where

$$P_{2,R} = \sqrt{P_{2,T}^2 + P_{2,V}^2} \quad (63)$$

and

$$\psi_2|_{\phi=\phi_1} = \arcsin \left(\frac{1}{k_2} \cdot \sqrt{\frac{1 + \sin(\phi_1 + \beta_2)}{2}} \right) \quad (64)$$

The moment anywhere along Segment 1 can be computed:

$$M(s) = P_{2,T} \cdot (a + b - y) + P_{1,T} \cdot (a - y) + P_{2,V} \cdot (\delta_{x1} + \delta_{x2} - x) + P_{1,V} \cdot (\delta_{x1} - x) \quad (65)$$

Differentiate both sides of (65) with respect to s to obtain:

$$\begin{aligned}\frac{dM}{ds} &= -(P_{2,T} + P_{1,T}) \cdot \frac{dy}{ds} - (P_{2,V} + P_{1,V}) \cdot \frac{dx}{ds} \\ &= -(P_{2,T} + P_{1,T}) \cdot \cos \phi - (P_{2,V} + P_{1,V}) \cdot \sin \phi\end{aligned}\quad (66)$$

Substitute (66) into (2) to get:

$$EI_1 \cdot \frac{d^2\phi}{ds^2} + (P_{2,T} + P_{1,T}) \cdot \cos \phi + (P_{2,V} + P_{1,V}) \cdot \sin \phi = 0 \quad (67)$$

After multiplying both sides of (67) by $d\phi/ds$, we can rewrite the equation as follows:

$$\frac{d}{ds} \left[\frac{1}{2} \cdot EI_1 \cdot \left(\frac{d\phi}{ds} \right)^2 + (P_{2,T} + P_{1,T}) \cdot \sin \phi - (P_{2,V} + P_{1,V}) \cdot \cos \phi \right] = 0 \quad (68)$$

From here, we can define the differentiation constant C as:

$$C = \frac{1}{2} \cdot EI_1 \cdot \left(\frac{d\phi}{ds} \right)^2 + (P_{2,T} + P_{1,T}) \cdot \sin \phi - (P_{2,V} + P_{1,V}) \cdot \cos \phi \quad (69)$$

To solve for C , we consider (14), and derive a similar equation:

$$\frac{d\phi}{ds} = \sqrt{\frac{2}{EI_2}} \cdot \sqrt{P_{2,R} \cdot \sin(\phi_2 + \beta_2) - P_{2,R} \cdot \sin(\phi + \beta_2)} \quad \text{for } \phi_1 \leq \phi \leq \phi_2 \quad (70)$$

Substitute $\phi = \phi_1$ and $d\phi/ds = d\phi/ds|_{\phi=\phi_1}$ into (69) and solving, we find that the constant C is:

$$C = P_{2,R} \cdot \sin(\phi_2 + \beta_2) + P_{1,R} \cdot \sin(\phi_1 + \beta_1) \quad (71)$$

where $P_{1,R}$ and β_1 is given by:

$$P_{1,R} = \sqrt{P_{1,T}^2 + P_{1,V}^2} \quad (72)$$

$$\beta_1 = \arcsin \left(\frac{-P_{1,V}}{\sqrt{P_{1,T}^2 + P_{1,V}^2}} \right) \quad (73)$$

Substitute the value for C back into the equation and solving, we obtain:

$$\frac{d\phi}{ds} = \sqrt{\frac{2}{EI_1}} \cdot \sqrt{P_{2,R} \cdot \sin(\phi_2 + \beta_2) + P_{1,R} \cdot \sin(\phi_1 + \beta_1) - P_{2,R} \cdot \sin(\phi + \beta_2) - P_{1,R} \cdot \sin(\phi + \beta_1)} \quad (74)$$

Using trigonometric manipulation, the latter two terms can be rewritten as:

$$P_{2,R} \cdot \sin(\phi + \beta_2) + P_{1,R} \cdot \sin(\phi + \beta_1) = A \cdot \sin \left(\phi - \frac{\pi}{2} - \theta \right) \quad (75)$$

where

$$A = \sqrt{P_{2,R}^2 + P_{1,R}^2 + 2 \cdot P_{2,R} \cdot P_{1,R} \cdot \cos(\beta_2 - \beta_1)} \quad (76)$$

$$\theta = \arctan\left(\frac{P_{2,R} \cdot \cos \beta_2 + P_{1,R} \cdot \cos \beta_1}{P_{2,R} \cdot \sin \beta_2 + P_{1,R} \cdot \sin \beta_1}\right) \quad (77)$$

We can now manipulate (74) to include A :

$$\frac{d\phi}{ds} = \sqrt{\frac{2 \cdot A}{EI_1}} \cdot \sqrt{\frac{P_{2,R}}{A} \cdot \sin(\phi_2 + \beta_2) + \frac{P_{1,R}}{A} \cdot \sin(\phi_1 + \beta_1) - \sin\left(\phi - \frac{\pi}{2} - \theta\right)} \quad (78)$$

Now we find L_1 by solving (78) for ds and then integrating ds from 0 to ϕ_1 :

$$\begin{aligned} L_1 &= \int_0^{\phi_1} ds \\ &= \sqrt{\frac{EI_1}{2 \cdot A}} \cdot \int_0^{\phi_1} \frac{d\phi}{\sqrt{\frac{P_{2,R}}{A} \cdot \sin(\phi_2 + \beta_2) + \frac{P_{1,R}}{A} \cdot \sin(\phi_1 + \beta_1) - \sin\left(\phi - \frac{\pi}{2} - \theta\right)}} \end{aligned} \quad (79)$$

So we let:

$$\sin^2 \psi = \frac{1 + \sin\left(\phi - \frac{\pi}{2} - \theta\right)}{1 + \frac{P_{2,R}}{A} \cdot \sin(\phi_2 + \beta_2) + \frac{P_{1,R}}{A} \cdot \sin(\phi_1 + \beta_1)} \quad (80)$$

Solving (80), we get:

$$\frac{P_{2,R}}{A} \cdot \sin(\phi_2 + \beta_2) + \frac{P_{1,R}}{A} \cdot \sin(\phi_1 + \beta_1) = \frac{1 + \sin\left(\phi - \frac{\pi}{2} - \theta\right)}{\sin^2} - 1 \quad (81)$$

Substituting into (81) into (79) and rearranging, we get:

$$L_1 = \sqrt{\frac{EI_1}{2 \cdot A}} \cdot \int_0^{\phi_1} \frac{\sin \psi \cdot d\phi}{\sqrt{\left[1 + \sin\left(\phi - \frac{\pi}{2} - \theta\right)\right] \cdot (1 - \sin^2 \psi)}} \quad (82)$$

Let:

$$k_1^2 \cdot \sin^2 \psi = \frac{1 + \sin\left(\phi - \frac{\pi}{2} - \theta\right)}{2} \quad (83)$$

Solving (83) gives us:

$$\sin\left(\phi - \frac{\pi}{2} - \theta\right) = 2 \cdot k_1^2 \cdot \sin^2 \psi - 1 \quad (84)$$

Substituting (84) into (81) gives:

$$\frac{P_{2,R}}{A} \cdot \sin(\phi_2 + \beta_2) + \frac{P_{1,R}}{A} \cdot \sin(\phi_1 + \beta_1) = 2 \cdot k_1^2 - 1 \quad (85)$$

Substituting (61) into (85) gives a governing equation:

$$\frac{P_{2,R}}{A} \cdot (2k_2^2 - 1) + \frac{P_{1,R}}{A} \cdot \sin(\phi_1 + \beta_1) = 2 \cdot k_1^2 - 1 \quad (86)$$

Additionally, by applying the Pythagorean Theorem to (84), we get:

$$\cos\left(\phi - \frac{\pi}{2} - \theta\right) = 2 \cdot k_1 \cdot \sin \psi \cdot \sqrt{1 - k_1^2 \cdot \sin^2} \quad (87)$$

Differentiating both sides of (87) gives:

$$\begin{aligned} \cos\left(\phi - \frac{\pi}{2} - \theta\right) \cdot d\phi &= 2 \cdot k_1^2 \cdot 2 \cdot \sin \psi \cdot \cos \psi \cdot d\psi \\ d\phi &= \frac{2 \cdot k_1 \cdot \cos \psi \cdot d\psi}{\sqrt{1 - k_1^2 \cdot \sin^2}} \end{aligned} \quad (88)$$

Substituting into the equation for L_1 , we get:

$$L_1 = \sqrt{\frac{EI_1}{A}} \cdot \int_0^{\phi_1} \frac{d\psi}{\sqrt{1 - k_1^2 \cdot \sin^2}} \quad (89)$$

Now we need to change the limits of integration. Substituting $\phi = \phi_1$ into (84) and solving for ψ , the upper limit of integration becomes:

$$\psi_1|_{\phi=\phi_1} = \arcsin\left[\frac{1}{\sqrt{2} \cdot k_1} \cdot \sqrt{1 + \sin\left(\phi_1 - \frac{\pi}{2} - \theta\right)}\right] \quad (90)$$

Substituting $\phi = 0$ into (84) and solving for ψ , the lower limit of integration becomes:

$$\psi_1|_{\phi=0} = \arcsin\left[\frac{1}{\sqrt{2} \cdot k_1} \cdot \sqrt{1 + \sin\left(-\frac{\pi}{2} - \theta\right)}\right] \quad (91)$$

So, we can rewrite (89) as:

$$\begin{aligned} L_1 &= \sqrt{\frac{EI_1}{A}} \cdot \left[\int_0^{\psi_1(\phi=\phi_1)} \frac{d\psi}{\sqrt{1 - k_1^2 \cdot \sin^2}} - \int_0^{\psi_1(\phi=0)} \frac{d\psi}{\sqrt{1 - k_1^2 \cdot \sin^2}} \right] \\ &= \sqrt{\frac{EI_1}{A}} \cdot [F_n(\psi_1(\phi = \phi_1), k_1, 0) - F_n(\psi_1(\phi = 0), k_1, 0)] \end{aligned} \quad (92)$$

To solve for the tip angles ϕ_1 and ϕ_2 , we follow this process: 1) Guess a value for ϕ_1 , 2) Adjust k_1 until (92) is true, 3) Adjust k_2 until (62) is true, 4) Verify to see if (86) holds true. If (86) does not hold true, then a new guess for ϕ_1 will be required.

3.2. Solving for δ_{x1} , the Horizontal Deflection of Segment 1

We know $dx/ds = \sin\phi$ and (79). Solve for ds , substitute into dx/ds , and solve for dx :

$$\begin{aligned} dx &= ds \cdot \sin\phi \\ &= \frac{\sin\phi \cdot d\phi}{\sqrt{\frac{2A}{EI_1} \cdot \sqrt{\frac{P_{2,R}}{A} \cdot \sin(\phi_2 + \beta_2) + \frac{P_{1,R}}{A} \cdot \sin(\phi_1 + \beta_1) - \sin\left(\phi - \frac{\pi}{2} - \theta\right)}}} \end{aligned} \quad (93)$$

Now integrating from $\phi = 0$ to $\phi = \phi_1$ to get:

$$\delta_{x1} = \sqrt{\frac{EI_1}{2A}} \cdot \int_0^{\phi_1} \frac{\sin\phi \cdot d\phi}{\sqrt{\frac{P_{2,R}}{A} \cdot \sin(\phi_2 + \beta_2) + \frac{P_{1,R}}{A} \cdot \sin(\phi_1 + \beta_1) - \sin\left(\phi - \frac{\pi}{2} - \theta\right)}} \quad (94)$$

Perform the same substitutions as before to get into the elliptic integral format:

$$\delta_{x1} = \sqrt{\frac{EI_1}{A}} \cdot \int_{\psi_1(\phi=0)}^{\psi_1(\phi=\phi_1)} \frac{\sin\phi \cdot d\psi}{\sqrt{1 - k_1^2 \cdot \sin^2}} \quad (95)$$

To express $\sin\phi$ in terms of ψ , we take (84) and solve for ϕ :

$$\begin{aligned} \sin\left(\phi - \frac{\pi}{2} - \theta\right) &= 2 \cdot k_1^2 \cdot \sin^2 - 1 \\ \phi - \frac{\pi}{2} - \theta &= \arcsin(2 \cdot k_1^2 \cdot \sin^2 - 1) \\ \phi &= \arcsin(2 \cdot k_1^2 \cdot \sin^2 - 1) + \frac{\pi}{2} + \theta \end{aligned} \quad (96)$$

Taking the sine of both sides will yield:

$$\begin{aligned} \sin\phi &= \sin\left[\arcsin(2 \cdot k_1^2 \cdot \sin^2 - 1) + \left(\frac{\pi}{2} + \theta\right)\right] \\ &= (2 \cdot k_1^2 \cdot \sin^2 - 1) \cdot \cos\left(\frac{\pi}{2} + \theta\right) + \cos\left[\arcsin(2 \cdot k_1^2 \cdot \sin^2 - 1)\right] \cdot \sin\left(\frac{\pi}{2} + \theta\right) \\ &= (2 \cdot k_1^2 \cdot \sin^2 - 1) \cdot \cos\left(\frac{\pi}{2} + \theta\right) + 2 \cdot k_1 \cdot \sin\psi \cdot \sqrt{1 - k_1^2 \cdot \sin^2} \cdot \sin\left(\frac{\pi}{2} + \theta\right) \end{aligned} \quad (97)$$

Now, substituting back into our equation for δ_{x1} we get:

$$\begin{aligned}
\delta_{x1} &= \sqrt{\frac{EI_1}{A}} \cdot \int_{\psi_1(\phi=0)}^{\psi_1(\phi=\phi_1)} \frac{(2 \cdot k_1^2 \cdot \sin^2 \psi - 1) \cdot \cos\left(\frac{\pi}{2} + \theta\right) \cdot d\psi}{\sqrt{1 - k_1^2 \cdot \sin^2 \psi}} \\
&\quad + \sqrt{\frac{EI_1}{A}} \cdot \int_{\psi_1(\phi=0)}^{\psi_1(\phi=\phi_1)} 2 \cdot k_1 \cdot \sin \psi \cdot \sin\left(\frac{\pi}{2} + \theta\right) \cdot d\psi \\
&= \sqrt{\frac{EI_1}{A}} \cdot \cos\left(\frac{\pi}{2} + \theta\right) \cdot \int_{\psi_1(\phi=0)}^{\psi_1(\phi=\phi_1)} \frac{-2 \cdot (1 - k_1^2 \cdot \sin^2 \psi) + 1}{\sqrt{1 - k_1^2 \cdot \sin^2 \psi}} \cdot d\psi \\
&\quad - \sqrt{\frac{EI_1}{A}} \cdot \sin\left(\frac{\pi}{2} + \theta\right) \cdot 2 \cdot k_1 \cdot \cos \psi \Big|_{\psi_1(\phi=0)}^{\psi_1(\phi=\phi_1)} \\
&= -2 \cdot \sqrt{\frac{EI_1}{A}} \cdot \cos\left(\frac{\pi}{2} + \theta\right) \cdot \int_{\psi_1(\phi=0)}^{\psi_1(\phi=\phi_1)} \sqrt{1 - k_1^2 \cdot \sin^2 \psi} \cdot d\psi \\
&\quad + \sqrt{\frac{EI_1}{A}} \cdot \cos\left(\frac{\pi}{2} + \theta\right) \cdot \int_{\psi_1(\phi=0)}^{\psi_1(\phi=\phi_1)} \frac{d\psi}{\sqrt{1 - k_1^2 \cdot \sin^2 \psi}} \\
&\quad - \sqrt{\frac{EI_1}{A}} \cdot \sin\left(\frac{\pi}{2} + \theta\right) \cdot 2 \cdot k_1 \cdot \left[\cos(\psi_1(\phi = \phi_1)) - \cos(\psi_1(\phi = 0)) \right] \\
&= -2 \cdot \sqrt{\frac{EI_1}{A}} \cdot \cos\left(\frac{\pi}{2} + \theta\right) \cdot \left[E_n(\psi_1(\phi = \phi_1), k_1) - E_n(\psi_1(\phi = 0), k_1) \right] \\
&\quad + \sqrt{\frac{EI_1}{A}} \cdot \cos\left(\frac{\pi}{2} + \theta\right) \cdot \left[F_n(\psi_1(\phi = \phi_1), k_1, 0) - F_n(\psi_1(\phi = 0), k_1, 0) \right] \\
&\quad - \sqrt{\frac{EI_1}{A}} \cdot \sin\left(\frac{\pi}{2} + \theta\right) \cdot 2 \cdot k_1 \cdot \left[\cos(\psi_1(\phi = \phi_1)) - \cos(\psi_1(\phi = 0)) \right] \tag{98}
\end{aligned}$$

3.3. Solving for a , the Vertically Deflected Height of Segment 1

We know $dy/ds = \cos\phi$ and (79). Solving for dy , integrating from $\phi = 0$ to $\phi = \phi_1$, and performing substitutions, we get:

$$a = \sqrt{\frac{EI_1}{A}} \cdot \int_{\psi_1(\phi=0)}^{\psi_1(\phi=\phi_1)} \frac{\cos \phi \cdot d\psi}{\sqrt{1 - k_1^2 \cdot \sin^2 \psi}} \tag{99}$$

Now we express $\cos \phi$ in terms of ψ :

$$\begin{aligned}
\cos \phi &= \cos \left[\arcsin(2 \cdot k_1^2 \cdot \sin^2 \psi - 1) + \left(\frac{\pi}{2} + \theta\right) \right] \\
&= \cos \left[\arcsin(2 \cdot k_1^2 \cdot \sin^2 \psi - 1) \right] \cdot \cos\left(\frac{\pi}{2} + \theta\right) - (2 \cdot k_1^2 \cdot \sin^2 \psi - 1) \cdot \sin\left(\frac{\pi}{2} + \theta\right) \\
&= 2 \cdot k_1 \cdot \sin \psi \cdot \sqrt{1 - k_1^2 \cdot \sin^2 \psi} \cdot \cos\left(\frac{\pi}{2} + \theta\right) - (2 \cdot k_1^2 \cdot \sin^2 \psi - 1) \cdot \sin\left(\frac{\pi}{2} + \theta\right) \tag{100}
\end{aligned}$$

Substituting back into the equation for a , we get:

$$\begin{aligned}
a &= \sqrt{\frac{EI_1}{A}} \cdot \int_{\psi_1(\phi=0)}^{\psi_1(\phi=\phi_1)} 2 \cdot k_1 \cdot \sin \psi \cdot d\psi \cdot \cos\left(\frac{\pi}{2} + \theta\right) \\
&\quad - \sqrt{\frac{EI_1}{A}} \cdot \int_{\psi_1(\phi=0)}^{\psi_1(\phi=\phi_1)} \frac{-2 \cdot (1 - k_1^2 \cdot \sin^2 \psi) + 1}{\sqrt{1 - k_1^2 \cdot \sin^2 \psi}} \cdot d\psi \cdot \sin\left(\frac{\pi}{2} + \theta\right) \\
&= \sqrt{\frac{EI_1}{A}} \cdot \cos\left(\frac{\pi}{2} + \theta\right) \cdot 2k_1 \cdot (-\cos \psi) \Big|_{\psi_1(\phi=0)}^{\psi_1(\phi=\phi_1)} \\
&\quad - \sqrt{\frac{EI_1}{A}} \cdot \sin\left(\frac{\pi}{2} + \theta\right) \cdot (-2) \cdot \int_{\psi_1(\phi=0)}^{\psi_1(\phi=\phi_1)} \sqrt{1 - k_1^2 \cdot \sin^2 \psi} \cdot d\psi \\
&\quad - \sqrt{\frac{EI_1}{A}} \cdot \sin\left(\frac{\pi}{2} + \theta\right) \cdot \int_{\psi_1(\phi=0)}^{\psi_1(\phi=\phi_1)} \frac{1}{\sqrt{1 - k_1^2 \cdot \sin^2 \psi}} \cdot d\psi \\
&= -\sqrt{\frac{EI_1}{A}} \cdot \cos\left(\frac{\pi}{2} + \theta\right) \cdot 2k_1 \cdot \left[\cos(\psi_1(\phi = \phi_1)) - \cos(\psi_1(\phi = 0)) \right] \\
&\quad + 2 \cdot \sqrt{\frac{EI_1}{A}} \cdot \sin\left(\frac{\pi}{2} + \theta\right) \cdot \left[E_n(\psi_1(\phi = \phi_1), k_1) - E_n(\psi_1(\phi = 0), k_1) \right] \\
&\quad - \sqrt{\frac{EI_1}{A}} \cdot \sin\left(\frac{\pi}{2} + \theta\right) \cdot \left[F_n(\psi_1(\phi = \phi_1), k_1, 0) - F_n(\psi_1(\phi = 0), k_1, 0) \right] \tag{101}
\end{aligned}$$

3.4. Solving for δ_{x2} , the Horizontal Deflection of Segment 2

Based on the results of the single load case, we can write:

$$\begin{aligned}
\delta_{x2} &= -\sqrt{\frac{EI}{P_{2,R}}} \cdot 2 \cos \beta_2 \cdot \left[E_n\left(\frac{\pi}{2}, k_2\right) - E_n(\psi_2(\phi = \phi_1), k_2) \right] \\
&\quad + \sqrt{\frac{EI}{P_{2,R}}} \cdot \cos \beta_2 \cdot \left[F_n\left(\frac{\pi}{2}, k_2, 0\right) - F_n(\psi_2(\phi = \phi_1), k_2, 0) \right] \\
&\quad - \sqrt{\frac{EI}{P_{2,R}}} \cdot 2k_2 \cdot \sin \beta_2 \cdot \cos(\psi_2(\phi = \phi_1)) \tag{102}
\end{aligned}$$

3.5. Solving for b , the Vertically Deflected Height of Segment 2

Based on the results of the single load case, we can write:

$$\begin{aligned}
b = & -\sqrt{\frac{EI}{P_{2,R}}} \cdot 2 \sin \beta_2 \cdot \left[E_n \left(\frac{\pi}{2}, k_2 \right) - E_n(\psi_2(\phi = \phi_1), k_2) \right] \\
& + \sqrt{\frac{EI}{P_{2,R}}} \cdot \sin \beta_2 \cdot \left[F_n \left(\frac{\pi}{2}, k_2, 0 \right) - F_n(\psi_2(\phi = \phi_1), k_2, 0) \right] \\
& + \sqrt{\frac{EI}{P_{2,R}}} \cdot 2k_2 \cdot \cos \beta_2 \cdot \cos(\psi_2(\phi = \phi_1))
\end{aligned} \tag{103}$$

4. Validation: Comparing the Analytical Solution to FEA

As a validation step, we will compare the deflection angles calculated from the Analytical Solution to those calculated from Finite Element Analysis (FEA). We first approach the 1-element case shown in Figure 2, the results of which are given in Table 1. We then approach the 2-element case shown in Figure 3, the results of which are given in Table 2. In all cases, the tip angles calculated from the Analytical Solution are nearly identical to the same values calculated using FEA.

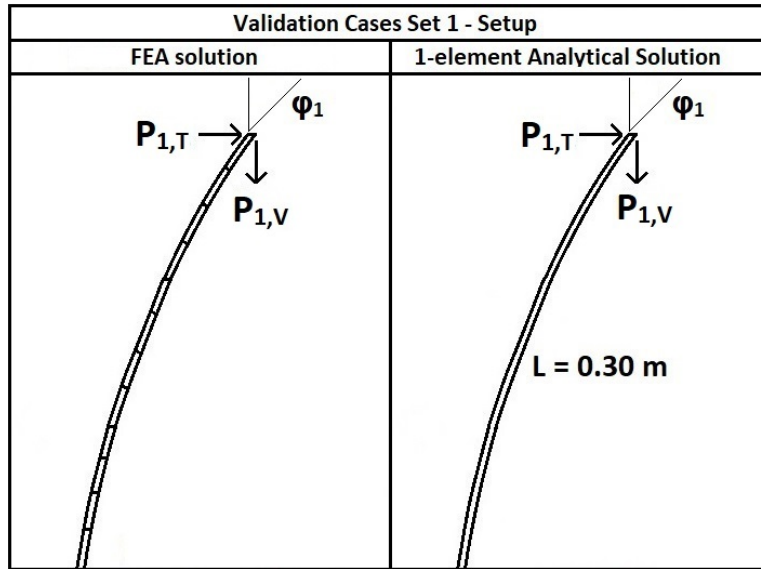


Figure 2: Loading conditions in validation cases set 1.

Table 1: 1-element Pole^a with a Tip Load^b: Analytical Solution versus FEA

Validation Cases Set 1	FEA	Analytical	
Point Load 1	ϕ_1	ϕ_1	k
$P_{T1} = 3.92, P_{V1} = 0$	36.34°	36.10°	0.89139
$P_{T1} = 3.92, P_{V1} = 0.2$	37.15°	36.91°	0.88291
$P_{T1} = 3.92, P_{V1} = 0.4$	38.00°	37.76°	0.87433
$P_{T1} = 3.92, P_{V1} = 1$	40.71°	40.49°	0.84887
$P_{T1} = 3.92, P_{V1} = 2$	45.87°	45.71°	0.81249
$P_{T1} = 3.92, P_{V1} = 3$	51.87°	51.80°	0.79002
$P_{T1} = 3.92, P_{V1} = 3.92$	58.09°	58.12°	0.78329
$P_{T1} = 3.92, P_{V1} = 6$	73.91°	74.07°	0.80504
$P_{T1} = 3.92, P_{V1} = 8$	89.37°	89.43°	0.84589
$P_{T1} = 3.92, P_{V1} = 9$	96.53°	96.49°	0.86612

^aThe pole has length of $L = 0.3$ m and flexural rigidity $EI = 0.24$.

^bAll point loads are in Newtons.

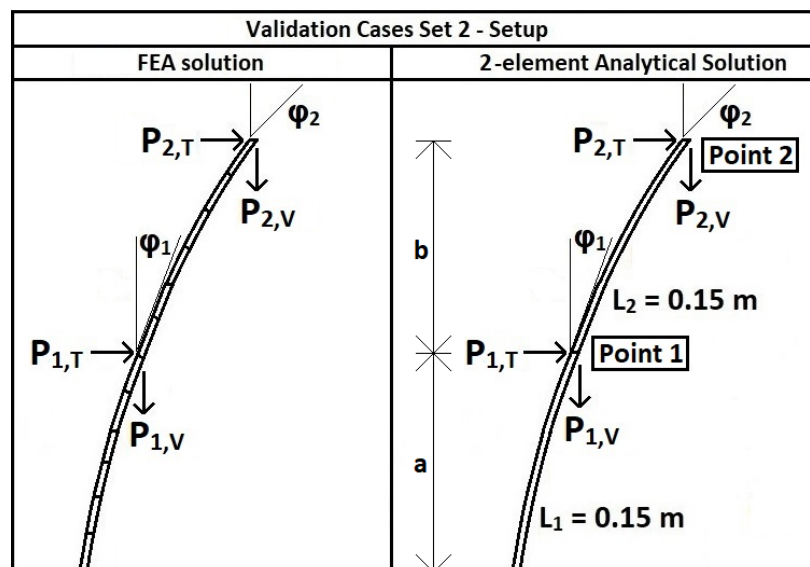


Figure 3: Loading conditions in validation cases set 2.

Table 2: 2-element Pole^a with 2 Tip Loads^b: Analytical Solution versus FEA

Validation Cases Set 2		FEA		Analytical			
Point Load 2	Point Load 1	ϕ_2	ϕ_1	ϕ_2	ϕ_1	k_2	k_1
$F_{T2} = 3.92, F_{V2} = 0$	$F_{T1} = 3.92, F_{V1} = 0$	43.09°	35.19°	42.80°	34.89°	0.91628	0.90174
$F_{T2} = 3, F_{V2} = 0$	$F_{T1} = 3, F_{V1} = 0$	35.35°	28.66°	35.11°	28.42°	0.88742	0.87328
$F_{T2} = 2, F_{V2} = 0$	$F_{T1} = 2, F_{V1} = 0$	25.18°	20.28°	25.13°	20.22°	0.84394	0.83220
$F_{T2} = 1, F_{V2} = 0$	$F_{T1} = 1, F_{V1} = 0$	13.20°	10.58°	13.20°	10.58°	0.78370	0.77644
$F_{T2} = 2, F_{V2} = 0$	$F_{T1} = 1, F_{V1} = 0$	22.89°	17.90°	22.80°	17.82°	0.83294	0.82368
$F_{T2} = 2, F_{V2} = 2$	$F_{T1} = 2, F_{V1} = 2$	35.84°	28.38°	36.24°	28.76°	0.65106	0.62624
$F_{T2} = 4, F_{V2} = 2$	$F_{T1} = 2, F_{V1} = 2$	52.36°	41.33°	52.29°	41.31°	0.84676	0.78858
$F_{T2} = 5, F_{V2} = 1$	$F_{T1} = 2, F_{V1} = 2$	52.61°	41.95°	52.36°	41.77°	0.91016	0.84248
$F_{T2} = 1, F_{V2} = 5$	$F_{T1} = 2, F_{V1} = 2$	48.88°	37.34°	50.96°	39.08°	0.51706	0.56990

^aSegments 1 and 2 have length of $L_1 = L_2 = 0.15$ m and flexural rigidity $EI = 0.24$.

^bAll point loads are in Newtons.

5. Conclusion

The research has presented a methodology for analyzing the large deformation of a cantilever, e.g. a pole, with point loads at two points along the cantilever (and where each point load is allowed to have both a horizontal and a vertical component of force). The research is novel because the analytical solution for more than one tip load along a cantilever has not been published. It is hoped that the methodology contained herein will be of benefit to the field of non-linear mechanics.

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