A probabilistic approach to the enumeration of bounded Motzkin paths via the gambler's ruin

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Abstract. We make connections between the well-known gambler's ruin problem and the enumeration of bounded Motzkin and Dyck paths. We start with a basic recurrence relation for a variation of the gambler's ruin that permits ties, derive explicit formulas for the corresponding probability generating functions, explain the correspondence between this ruin variation and Motzkin paths, and obtain algebraic and rational expressions for the generating functions that enumerate height-bounded Motzkin and Dyck paths.

1. Introduction. In the classic gambler's ruin problem, a gambler and an opponent square off in a tournament comprised of independent games. In each game a player wins or loses a point from the other player until one of them runs out of points and is "ruined". The problem goes back to the beginnings of probability itself [5] and inherently involves Catalan numbers and Dyck paths. We consider a slight twist to the problem—in an individual game the players can also tie with no points exchanged. This three-outcome problem involves lattice walks called *Motzkin paths*. For a sampling of recent articles about Motzkin paths, see [2, 3, 7, 8, 10]. Unlike these articles, we approach enumeration of various types of Motzkin paths from a purely probabilistic point of view. We start with a recurrence relation that models the gambler's ruin problem, derive generating functions for the probabilities for the gambler to be ruined, apply differentiation to obtain known results for the moments of this distribution, and finally make connections to Motzkin paths. Our main contribution is obtaining algebraic and rational formulas in Equations (5.2), (5.4), (5.5), and (5.7) for the generating functions of height-bounded Motzkin paths and Dyck paths. This approach uses only elementary notions of probability.

In section 2 we review Dyck paths, Motzkin paths, and the basics of generating functions. In section 3 we lay out recurrence relations governing the probabilities of the gambler's ruin and the associated generating functions. In sections 4 and 6 we obtain explicit formulas for the generating functions of the probability distributions of the gambler's ruin, where the opponent has either finite or infinite resources, and compute moments of these distributions. In section 5 we make connections between the gambler's ruin problem and Motzkin paths and derive new formulas for the generating functions of height-bounded Motzkin and Dyck paths. We conclude in section 7 with further possibilities for relating gambler's ruin variations with other kinds of restricted Motzkin paths.

2. Background. The central tool for analyzing sequences of combinatorial objects is the generating function. If $\langle a_k : k \geq 0\rangle$ is a sequence of numerical values, the generating function of the sequence (in the variable x) is the power series $\sum_{k\geq 0} a_k x^k$. There are two main methods

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of computing generating functions—the recursive method, examined for instance, in [11] and the symbolic method, for instance, in Part A of [4]. We use both. Generating functions are particularly useful in computing statistics. One basic result relates the moments of the probability distribution of a random variable to the derivatives of the generating function of the distribution. If X is a random variable with integer support and $P(x) = \sum_{k\geq 0} \text{prob}(X = k)x^k$ is the generating function of its probability distribution, then $P(1) = \sum_{k\geq 0} \text{prob}(X = k) = 1$, $E(X) = P'(1)$, and $\text{Var}(X) = P''(1) + P'(1) - \big(P'(1)\big)^2$.

The Catalan numbers, A000108 in [9], enumerate many combinatorial objects, including triangulated polygons, binary trees, strings of matched parentheses, and lattice paths known as Dyck paths, which are helpful tools in visualizing the game play of the classic gambler's ruin. A Dyck path starts at the origin, remains on or above the x-axis, consists of up steps with displacement $(1, 1)$ and down steps with displacement $(1, -1)$, and ends back on the x-axis. The kth Catalan number c_k is the number of distinct Dyck paths ending at $(2k, 0)$, where k is commonly referred to as the semi-length of the path. A cousin of the Dyck path is the Motzkin path, which has the same parameters as a Dyck path, but with an additional allowance of a horizontal step with displacement $(1, 0)$. The number m_k of distinct Motzkin

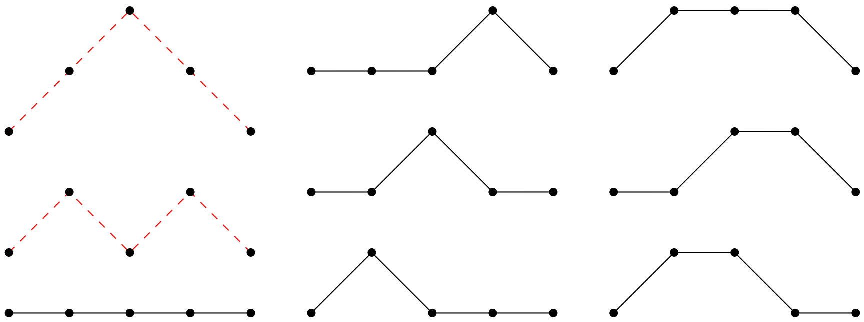

Figure 1. *All nine Motzkin paths of length four. The red dashed paths are also Dyck paths.*

paths ending at $(k, 0)$ is the kth Motzkin number, A001006 in [9]. Figure 1 shows that $c_2 = 2$ and $m_4 = 9$.

Often, steps are weighted with values representing probabilities or counts. We weight up, down, and horizontal steps with the values p, q, and r, respectively. The weight of a path is the product of the weights of the individual steps along the way. We do not specifically mention weights in our notation, and so m_k variably denotes either the *number* of Motzkin paths of length k when $p = q = r = 1$, the *number* of Dyck paths of semi-length k when $p = q = 1$ and $r = 0$, or the *probability* of traversing a Motzkin path of length k when p, q, r represent the probabilities of individual steps. Whenever the weights play a crucial role in a computation, we carefully specify them within context.

The symbolic method makes computing the generating function of weighted Motzkin paths almost effortless. A Motzkin path can be empty, a horizontal step followed by another Motzkin path, or an up step followed by a Motzkin path and a down step followed by another Motzkin path, as the black paths in Figure 2 show. Therefore, the generating function $M(x) = \sum_k m_k\, x^k$ for the weighted Motzkin numbers is a solution of $M = 1 + r\,xM + pq\,x^2M^2$, which

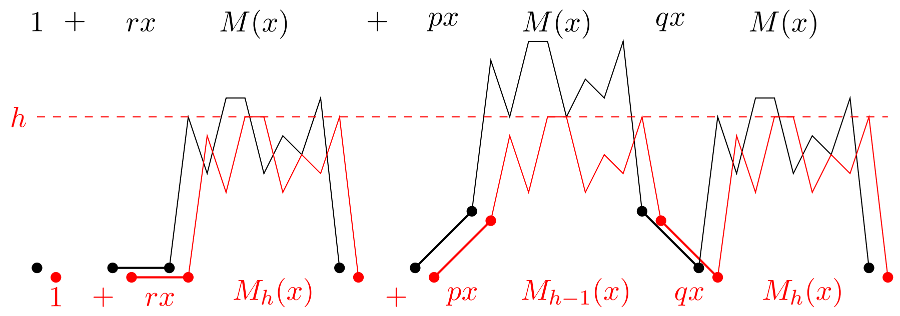

Figure 2. *Symbolic method derivation of the generating functions for unbounded Motzkin numbers (black) and bounded Motzkin numbers (red).*

has the two roots

$$M_{\pm}(x) = \frac{1 - rx \pm \sqrt{1 - 2rx + (r^2 - 4pq)x^2}}{2pq\,x^2}. \tag{2.1}$$

Since there is only one empty Motzkin path and one is a finite number, $M(x)$ must be the negative root $M_{-}(x)$. We maintain the distinction between the negative and positive roots for referencing in later equations.

Similarly, the generating function $C(x) = \sum_k c_k x^k$ for the weighted Catalan numbers is a root of the quadratic $C = 1 + pq\,xC^2$, which results from application of the symbolic method on Dyck paths (keeping in mind that only up steps carry an x term in a Dyck path). The two roots of this equation are

$$C_{\pm}(x) = \frac{1 \pm \sqrt{1 - 4pq\,x}}{2pq\,x}, \tag{2.2}$$

where $C_{-}(x)$ is equivalent to $C(x)$.

We also consider Dyck paths and Motzkin paths with a height restriction and define $c_{h,k}$ to be the number of Dyck paths of semi-length k that have a maximum height of at most h and $m_{h,k}$ to be the number of Motzkin paths of length k that have a maximum height of at most h. Figure 1 shows that $c_{0,2} = 0$, $c_{1,2} = 1$, and $c_{2,2} = 2$, while $m_{0,4} = 1$, $m_{1,4} = 8$, and $m_{2,4} = 9$. A Motzkin path of height at most h can be empty, a horizontal step followed by a Motzkin path of height at most h, or an up step followed by a Motzkin path of height at most $h-1$ and a down step followed by another Motzkin path of height at most h, as the red paths in Figure 2 show. Again by the symbolic method, the generating function $M_h(x) = \sum_k m_{h,k} x^k$ must satisfy $M_h(x) = 1 + r\,xM_h(x) + pq\,x^2 M_{h-1}(x)M_h(x)$. Solving for $M_h(x)$ results in the recurrence

$$M_h(x) = \frac{1}{1 - r\,x - pq\,x^2 M_{h-1}(x)} \tag{2.3}$$

for $h \geq 1$. Since there is only one path of height 0 for any length and it consists entirely of horizontal steps, $M_0(x) = 1/(1 - rx)$. Repeated application of Equation (2.3) gives the first

h levels of the continued fraction

$$M_h(x) = \cfrac{1}{1 - rx - \cfrac{pq\,x^2}{1 - rx - \cfrac{pq\,x^2}{\ddots \over 1 - rx - \cfrac{pq\,x^2}{1 - rx}}}}. \tag{2.4}$$

Equation (2.4) deserves several comments. First of all, it is well-known and appears, for example, as Theorem 10.9.1 in [2], Proposition 5 in [3], Example V.21 in [4], and Equation 25 in [10]. We derive new representations of $M_h(x)$ as an algebraic function in Equation (5.2) and as a rational function in Equation (5.4). It is a quick exercise to show the full continued fraction in Equation (2.4) gives the generating function for the unbounded Motzkin numbers, $M_{-}(x)$, in Equation (2.1). Lastly, many of the sequences of height-bounded Motzkin paths appear in [9] with $M_1(x), \ldots, M_6(x)$ as A011782, A171842, A005207, A094286, A094287, A094288, respectively, with Alois P. Heinz commenting on their enumeration of height-bounded Motzkin paths.

For the sake of completeness, we also mention $C_h(x)$, the generating function of bounded Catalan numbers, which satisfies the recurrence

$$C_h(x) = \frac{1}{1 - pq\,xC_{h-1}(x)} \tag{2.5}$$

for $h \geq 1$ with the initial condition $C_0(x) = 1$. This recurrence appears as Corollary 3 in [6], and its repeated application produces the continued fraction

$$C_h(x) = \cfrac{1}{1 - \cfrac{pq\,x}{1 - \cfrac{pq\,x}{\ddots \over 1 - \cfrac{}{1 - pq\,x}}}}, \tag{2.6}$$

which can also be obtained from making the substitutions of 0 for r and $\sqrt{x}$ for x in Equation (2.4). Once again, many of the sequences for height-bounded Dyck paths appear as individual entries in [9] for different values of h, but A080934 combines them all into a single table.

3. The general gambler's ruin and recurrence relations. In modeling the tournament of independent games between the gambler and the opponent, we assume the gambler begins with finite resources, say a tally of i points. In each game the gambler wins one point from the opponent with probability p, loses one with probability q, and ties with probability r. The games continue until one player runs out of points, if it ends at all. To quantify the tournament, we make several definitions. Let GR_i be the event that the gambler is ruined before the opponent and p_i be the probability for this to happen, i.e., $p_i = \text{prob}(GR_i)$. In this case, let the random variable N_i be the number of games for the gambler to be ruined, and $p_{i,t}$ be the probability that $N_i = t$, i.e., $p_{i,t} = \text{prob}(N_i = t)$. For the gambler to be ruined, it must happen in some number of games, so $p_i = \sum_{t\geq 0} p_{i,t}$. For each $i \geq 0$, we also define the generating function $P_i(x) = \sum_{t\geq 0} p_{i,t}x^t$.

A general recurrence relation follows from conditioning on the three possible outcomes of the first game. After the first game, the tournament can be thought of as starting over with the gambler's updated tally of points from a win, loss, or tie with the opponent, but one less game in which to be ruined. Therefore, a recurrence relation for the probabilities is

$$p_{i,t} = p\, p_{i+1,t-1} + r\, p_{i,t-1} + q\, p_{i-1,t-1} \text{ for all } i, t \geq 1. \tag{3.1}$$

There are boundary conditions for starting with no points and no games remaining. First, $p_{0,0} = 1$ and $p_{0,t} = 0$ for $t \geq 1$ because if the gambler starts without any points, then he immediately meets his ruin. It follows that $P_0(x) = 1$. If no games remain and the gambler has a positive tally of points, then the gambler has yet to be ruined and $p_{i,0} = 0$ for $i \geq 1$. We discuss further boundary conditions in sections 4 and 6.

A similar recurrence holds for the generating functions by multiplying the terms of Equation (3.1) by x^t and summing over $t \geq 1$, resulting in

$$P_i(x) = pxP_{i+1}(x) + rxP_i(x) + qxP_{i-1}(x) \text{ for all } i \geq 1. \tag{3.2}$$

For each x, Equation (3.2) is a recurrence in the variable i for the values of the generating function $P_i(x)$. This recurrence is linear, second order, homogeneous, and has "constant" coefficients ("constant" meaning that they do not depend on i, but very well could depend on x, p, q, r). The characteristic polynomial of the recurrence in Equation (3.2), which Lengyel also obtains in [8], is $(px)\rho^2 + (rx - 1)\rho + qx$. Solving for the roots results in

$$\rho_\pm = \frac{1 - rx \pm \sqrt{1 - 2rx + (r^2 - 4pq)x^2}}{2px}, \tag{3.3}$$

where we have suppressed the dependence in $\rho_\pm$ on x, p, q, r for clarity and brevity's sake. The solution to the recurrence in Equation (3.2) must have the form

$$P_i(x) = A_+(x)\rho_+^i + A_-(x)\rho_-^i \text{ for all } i \geq 0, \tag{3.4}$$

where $A_\pm(x)$ are functions of x that can be determined from boundary conditions.

For comparison with a well-known example, consider the Fibonacci numbers $\langle f_i : i \geq 0\rangle = 0, 1, 1, 2, 3, 5, \ldots$, which also satisfy a linear, second-order, homogeneous recurrence with constant coefficients, namely $f_i = f_{i-1} + f_{i-2}$. Binet's formula

$$f_i = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^i - \left(\frac{1-\sqrt{5}}{2}\right)^i\right) \tag{3.5}$$

gives an explicit formula for the Fibonacci numbers and is also a linear combination of powers of the two roots of the characteristic polynomial $\rho^2 - \rho - 1$ of the recurrence. Oddly, it is not immediately obvious from Equation (3.5) that f_i is even a rational number, much less an integer. However, just as in Identity 235 in [1] which is attributed to Catalan, we can use the binomial theorem to expand the powers of the characteristic roots and then cancel like terms to get a version of Binet's formula that obviously has a rational result

$$f_i = \frac{1}{2^{i-1}} \sum_{j=0}^{\lfloor \frac{i-1}{2} \rfloor} \binom{i}{2j+1} 5^j. \tag{3.6}$$

The main difference between these computations is that Binet's formula in Equation (3.5) gives an explicit formula for *numbers*, while Equation (3.4) gives an explicit formula for *generating functions*.

4. An opponent with finite resources and bounded Motzkin paths. Suppose the gambler and the opponent both begin the tournament with finite resources, say with a combined total of $n > 2$ points. To highlight the finiteness of the opponent's resources, in this section we append a subscript of n, now $P_{i,n}(x)$, to the probability generating function in Equation (3.4). As noted before, if the gambler starts with no points, he is immediately ruined and so $P_{0,n}(x) = 1$. On the other hand, if the gambler starts with all n points, he immediately wins, has no chance of losing at all, and so $P_{n,n}(x) = 0$. With these boundary conditions, we can evaluate Equation (3.4) at $i = 0$ and $i = n$ to solve for the coefficients, resulting in $A_+(x) = \rho_-^n/(\rho_-^n - \rho_+^n)$ and $A_-(x) = -\rho_+^n/(\rho_-^n - \rho_+^n)$. Plugging these solutions into Equation (3.4) yields the probability generating function

$$P_{i,n}(x) = \frac{\rho_-^n \rho_+^i - \rho_+^n \rho_-^i}{\rho_-^n - \rho_+^n}.$$

The product $\rho_+\rho_-$ of the roots of the characteristic polynomial is q/p, which we abbreviate as β, and so this equation simplifies to

$$P_{i,n}(x) = \beta^i \left(\frac{\rho_-^{n-i} - \rho_+^{n-i}}{\rho_-^n - \rho_+^n} \right). \tag{4.1}$$

From here, deriving formulas for the distribution and statistics of N_i is straightforward, simply by evaluating the generating function in Equation (4.1) and its derivatives at $x = 1$. For the case where $p = q$ the formulas result by taking the limit as $\beta \to 1$. For instance, the probability of the gambler's ruin starting with a tally of i is

$$p_i = P_{i,n}(1) = \begin{cases} \frac{\beta^i - \beta^n}{1-\beta^n} & \text{if } p \neq q \\ \frac{n-i}{n} & \text{if } p = q. \end{cases}$$

Likewise, the expected number of games for the gambler to be ruined, assuming he is the one to be ruined, is

$$E(N_i|GR_i) = \frac{P'_{i,n}(1)}{P_{i,n}(1)} = \begin{cases} \frac{2n(\beta^{n-i}-\beta^n) - i(1+\beta^{n-i}-\beta^n-\beta^{2n-i})}{(q-p)(1-\beta^{n-i})(1-\beta^n)} & \text{if } p \neq q \\ \frac{2ni-i^2}{6p} & \text{if } p = q. \end{cases}$$

The conditional variance, skewness, and further moments theoretically can be obtained by evaluating higher order derivatives of $P_{i,n}(x)$, but are behemoths of equations and consequently are omitted for the sake of brevity. However, with the help of symbolic programming, they are obtainable by continuing this process.

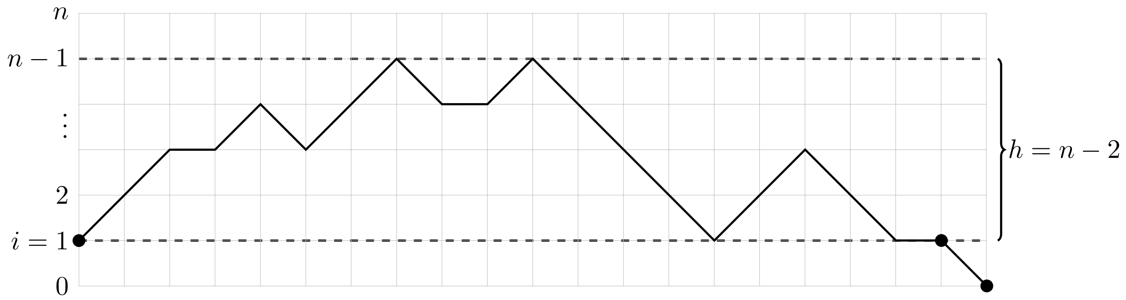

Figure 3. *A Motzkin path or tournament of a gambler starting with one point, followed by the final loss.*

5. Correspondence between lattice walks and tournaments. There is a direct correspondence between lattice walks and tournaments simply by equating up, down, and horizontal steps in a walk with wins, losses, and ties in a tournament. More specifically, there is a special correspondence between Motzkin paths and tournaments in which the gambler starts with exactly one point and eventually loses. In all but the last game, the gambler's tally must remain strictly positive, and in the penultimate game his tally must revert back to a single point, followed by a loss in the final game. In other words, a tournament for the gambler's ruin starting from one point traces out a Motzkin path, followed by a down step. If, in addition, the gambler and opponent start with a combined finite tally of n points, as discussed in section 4, the gambler's tally throughout the tournament must remain at or below $n-1$ points, else he would win. So, his ruin traces out a Motzkin path, bounded by a height of $(n-1)-1 = n-2$, as depicted in Figure 3. In terms of generating functions, this means $P_{1,n}(x) = qxM_{n-2}(x)$ or, equivalently,

$$\begin{aligned} M_h(x) &= \frac{P_{1,h+2}(x)}{qx} \\ &= \frac{1}{px}\frac{\rho_-^{h+1}-\rho_+^{h+1}}{\rho_-^{h+2}-\rho_+^{h+2}} \\ &= \frac{1}{px}\frac{\left(\frac{1-rx-\sqrt{1-2rx+(r^2-4pq)x^2}}{2px}\right)^{h+1}-\left(\frac{1-rx+\sqrt{1-2rx+(r^2-4pq)x^2}}{2px}\right)^{h+1}}{\left(\frac{1-rx-\sqrt{1-2rx+(r^2-4pq)x^2}}{2px}\right)^{h+2}-\left(\frac{1-rx+\sqrt{1-2rx+(r^2-4pq)x^2}}{2px}\right)^{h+2}} \end{aligned} \tag{5.1}$$

giving the generating function of the weighted and bounded Motzkin paths as an algebraic function. With a little more algebra, we can express $M_h(x)$ in terms of the two Motzkin roots

$M_{\pm}(x)$, abbreviated as $M_{\pm}$, from Equation (2.1) to get

$$M_h(x) = \frac{1}{pqx^2} \frac{M_-^{h+1} - M_+^{h+1}}{M_-^{h+2} - M_+^{h+2}}. \tag{5.2}$$

Amazingly, the *bounded* Motzkin paths can be generated from an algebraic combination of the generating function $M_-(x)$ of the *unbounded* Motzkin paths and its conjugate $M_+(x)$.

To get the generating function $M_h(x)$ for *counts* of bounded Motzkin paths, we simply make the substitutions $p = q = r = 1$ in Equation (5.2), giving

$$M_h(x) = 2\,\frac{\left(1 - x - \sqrt{1 - 2x - 3x^2}\right)^{h+1} - \left(1 - x + \sqrt{1 - 2x - 3x^2}\right)^{h+1}}{\left(1 - x - \sqrt{1 - 2x - 3x^2}\right)^{h+2} - \left(1 - x + \sqrt{1 - 2x - 3x^2}\right)^{h+2}}. \tag{5.3}$$

We can now represent $M_h(x)$ as a rational function, using the same process of expanding powers with the binomial theorem and cancelling like terms, as in Equations (3.5) and (3.6) that represent the Fibonacci numbers as rational numbers, resulting in

$$M_h(x) = \frac{2\sum_{j=0}^{\lfloor h/2 \rfloor} \binom{h+1}{2j+1} (1-x)^{h-2j} \left(1 - 2x - 3x^2\right)^j}{\sum_{j=0}^{\lfloor (h+1)/2 \rfloor} \binom{h+2}{2j+1} (1-x)^{h+1-2j} \left(1 - 2x - 3x^2\right)^j}. \tag{5.4}$$

This rational function is equivalent to the continued fraction in Equation (2.4). In Equation 10.74 of [2] Krattenthaler gives yet another representation of $M_h(x)$ as a ratio of Chebyshev polynomials of the second kind. Furthermore, making the substitution $r = 0$ into Equation (5.1) yields the generating function that enumerates height-bounded Motzkin paths without horizontal steps, i.e., height-bounded Dyck paths. Because of the caveat that Dyck paths are counted based on semi-length, the additional substitution of $\sqrt{x}$ for x is made to obtain the generating function for height-bounded Catalan numbers

$$\begin{aligned} C_h(x) &= \frac{1}{pqx} \frac{\left(\frac{1-\sqrt{1-4pqx}}{2pqx}\right)^{h+1} - \left(\frac{1+\sqrt{1-4pqx}}{2pqx}\right)^{h+1}}{\left(\frac{1-\sqrt{1-4pqx}}{2pqx}\right)^{h+2} - \left(\frac{1+\sqrt{1-4pqx}}{2pqx}\right)^{h+2}} \\ &= \frac{1}{pqx} \frac{C_-^{h+1} - C_+^{h+1}}{C_-^{h+2} - C_+^{h+2}}, \end{aligned} \tag{5.5}$$

once again, expressed in terms of the roots $C_{\pm}(x)$ in Equation (2.2). The additional substitutions of $p = q = 1$ in Equation (5.5) yield $C_h(x)$ for the *counts* of bounded Dyck paths as the algebraic function

$$C_h(x) = 2\,\frac{\left(1 - \sqrt{1-4x}\right)^{h+1} - \left(1 + \sqrt{1-4x}\right)^{h+1}}{\left(1 - \sqrt{1-4x}\right)^{h+2} - \left(1 + \sqrt{1-4x}\right)^{h+2}}. \tag{5.6}$$

Yet again, expanding the powers and canceling like terms gives

$$C_h(x) = \frac{2\displaystyle\sum_{j=0}^{\lfloor h/2\rfloor}\binom{h+1}{2j+1}(1-4x)^j}{\displaystyle\sum_{j=0}^{\lfloor (h+1)/2\rfloor}\binom{h+2}{2j+1}(1-4x)^j}. \tag{5.7}$$

6. An opponent with infinite resources and unbounded Motzkin paths. Now suppose the gambler squares off against an opponent, such as a casino, whose initial point tally dwarfs the gambler's. In this case, the gambler essentially has no chance of winning, and losing takes at least as many games as his initial tally. Therefore, $\lim_{i\to\infty} P_i(x) = 0$ for any x. The boundary condition for the ruin $P_0(x) = 1$, as well as the recurrence for the $P_i(x)$'s in Equation (3.2), still hold. To obtain an explicit formula for each $P_i(x)$, we must determine the coefficients $A_\pm(x)$ in Equation (3.4), but this requires an argument. Basic methods from calculus show that $\lim_{x\to 0}\rho_+ = \infty$ and $\lim_{x\to 0}\rho_- = 0$, and so there must be an interval about 0 where $\rho_+ > 1$ and $\rho_- < 1$ for each x in this interval. Taking the limits of the terms in Equation (3.4) as $i \to \infty$, we have

$$\lim_{i\to\infty} P_i(x) = 0 = \lim_{i\to\infty}\left[A_+(x)\rho_+^i + A_-(x)\rho_-^i\right] = A_+(x)\cdot\infty + A_-(x)\cdot 0 = A_+(x)\cdot\infty,$$

which forces $A_+(x) = 0$. The initial condition $P_0(x) = 1 = A_-(x)\rho_-^0$ then forces $A_-(x) = 1$ for x in this interval. However, as long as $A_+(x)$ and $A_-(x)$ are analytic, then it must be the case that $A_+(x) = 0$ and $A_-(x) = 1$ not just for x in this interval, but for *all* x in the common domains of the coefficients. Therefore, the generating function for the probabilities of the gambler's ruin when starting with i points is remarkably simple

$$P_i(x) = \rho_-^i \tag{6.1}$$

for all x. This time the terms of $P_1(x)$ represent *unbounded* Motzkin paths with up, down, and horizontal steps of weights p, q, r, respectively, followed by a final down step to finish off the ruin, and so $P_1(x) = M(x)qx$. This provides a completely probabilistic derivation of the fact that the generating function of the unbounded Motzkin paths is $M(x) = \rho_-/qx$, in agreement with $M_-(x)$ in Equation (2.1).

With the simpler generating function in hand, the computation of the probabilities and statistics for the distribution of the ruin is even easier. Recalling $\beta = q/p$, the probability of eventual ruin is

$$p_i = P_i(1) = \begin{cases}\beta^i & \text{if } p > q\\ 1 & \text{if } p \le q.\end{cases}$$

Again, the expected number of games for the gambler to be ruined, assuming he is the one to be ruined, is

$$E(N_i|GR_i) = \frac{P_i'(1)}{p_i} = \begin{cases}\frac{i}{|p-q|} & \text{if } p \ne q\\ \infty & \text{if } p = q.\end{cases}$$

and the conditional variance is

$$\mathrm{Var}(N_i|GR_i) = \frac{P_i''(1)}{p_i} + \frac{P_i'(1)}{p_i} - \left(\frac{P_i'(1)}{p_i}\right)^2 = \begin{cases} \frac{i(p+q-(p-q)^2)}{|p-q|^3} & \text{if } p \neq q \\ \infty & \text{if } p = q. \end{cases}$$

As before, higher order derivatives aid in finding further moments.

7. Further considerations. Our main result shows how the generating function for the gambler starting with a single point can be used to derive the generating function for height-bounded Motzkin paths. This methodology can be extended to derive the generating functions for other types of restricted Motzkin paths. For example, increasing the gambler's initial point tally to an arbitrary value corresponds to the Motzkin paths that have a minimum number of initial up-steps. Similarly, the gambler's final point tally corresponds to generating functions for Motzkin paths with a minimum number of final down-steps. Since the gambler either ends with all of the points or no points at all, there is a correspondence to Motzkin paths which either have a minimum final down-step requirement equivalent to their maximum allowable height or have no final down-step requirement at all. These types of restricted Motzkin paths are discussed in [10], which uses $M_{n|N}^{(a,b)}$ to denote the number of Motzkin paths of length n that start with at least a up steps, finish with at least b down steps, and have a maximum height of at most N.

Additionally, our main result starts with the gambling perspective and works towards the lattice-path enumeration. This process could be reversed by starting with some Motzkin path variation and deriving generating functions that describe a correlated variation of the gambler's ruin problem. For instance, a Motzkin path with an arbitrary minimum number of final down-steps corresponds to a gambling scenario in which one gambler has a lucky number of points for which he is repeatedly given the option to either cash-out or continue playing. Such a scenario may be difficult to analyze from merely the recurrence relation due to the ambiguous boundary conditions for the termination of the series of games, which attests to at least one benefit of starting from the lattice-path perspective.

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